New problem sheet (last!) on the web (hardcopy tomorrow)

I also put answers

Chap 11 HSD

Finish proof of Poincaré - Bendixson thm

ODE $\subset \mathbb{R}^2$. ω-limit sets are
- equilibria
- periodic solns
- network of equilibria and connecting orbits.

We showed before: * if $\omega(x)$ does not contain an equilibrium then $\Rightarrow$ $\omega(x)$ is a periodic soln.

Recall from the proof: we have shown that if $y \in \omega(x)$ and $z \in \omega(y)$ is not an eq.

$\Rightarrow \omega(y)$ is per. soln $\Rightarrow \omega(x)$ is per. soln.

It remains to consider the case that

$y \in \omega(x)$ and $z \in \omega(y)$ is an equilibrium.

$\Rightarrow$ $\omega(y)$ must be equilibrium and $\alpha(y)$ (taking argument of time backward) also must be equilibrium

$\Rightarrow$ either * y is itself an equilibrium

or * $y \in W^s(\omega(y)) \cap W^u(\alpha(y))$

$$\underset{\text{equilibrium}}{\uparrow} \qquad \underset{\text{equilibrium}}{\uparrow}$$

i.e. y lies on a connecting orbit between two equilibria

This resolves the two remaining cases. qed $\square$.

(This last part of the proof is not in HSD 10)

I will type this up (soon!).

Applications to biology

Infectious diseases (SIR and SIRS models)

In a simple model for the spread of diseases we divide the population into three groups:

S : susceptible to disease

I : infected

R : recovered.

for simplicity, we assume that the total population is constant.
(The latter implies that $\frac{d}{dt}(\underbrace{S+I+R}_{\text{total populatn}}) = 0$

Some more assumptions:

* once someone infected recovers this person cannot be infected again.
* the rate of transmission of disease is proportional to the both the number of infected and the number of susceptible people.

in a formula: $\frac{d}{dt} S = -\beta S \cdot I \qquad (\beta > 0)$

* the rate of recovery is assumed to be proportional to the number of infected individuals

in formulas: $\frac{d}{dt} I = \beta S \cdot I - \nu I \qquad (\nu > 0)$

$$\frac{d}{dt} R = \nu I$$

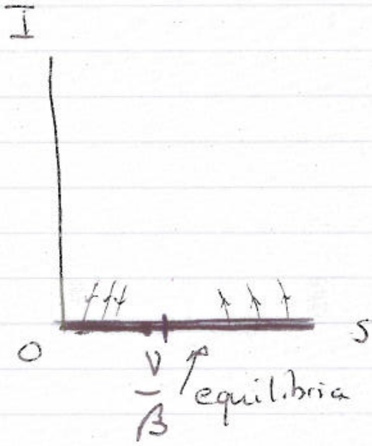
(one verifies that $\frac{d}{dt}(S+I+R) = 0$

Since the recovered population just follows from S and I it suffices to study the ODE in S and I (closed system).

$$\begin{cases} \dot{S} = -\beta S I \\ \dot{I} = \beta S I - \nu I \end{cases}$$

let us first compute the equilibria:

$$\begin{cases} I = 0 & \text{(i.e. } S\text{-axis)} \\ S = \dfrac{\nu}{\beta} \text{ and } I = 0 & \nearrow \text{already included} \end{cases}$$



to better understand the flow near the equilibria, we compute the derivative on $I=0$

$$\begin{pmatrix} 0 & -\beta S \\ 0 & \beta S - \nu \end{pmatrix}$$

$\Rightarrow$ one zero eigenvalue

(nb: if there was no zero eigenvalue, equilibria had to be isolated. but here we have an entire line of equilibria $\Rightarrow$ at least one zero eigenvalue with corresponding eigenvector pointing in the direction of this line.)

the other eigenvalue is $\beta S - \nu$: negative if $S < \frac{\nu}{\beta}$

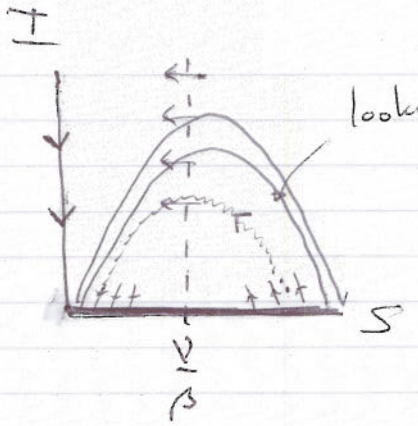
positive if $S > \frac{\nu}{\beta}$

$\exists$ one-dim unstable man

$\exists$ one-dim stable man

In order to get some grip on what's going on, it is sometimes useful to plot nullclines (lines on which a particular component of the vector field equals to zero) eg: $\dot{S} = 0 \Rightarrow S = 0$ or $I = 0$

(NB: not a systematic methodology) $\qquad \dot{I} = 0 \Rightarrow S = \frac{\nu}{\beta}$ or $I = 0$



$S = 0$ is flow-invariant subspace (since $\dot{S} = 0$ if $S = 0$)

Some relevant questions: * for which initial conditions we have

$$\lim_{t\to\infty} S(t) = \lim_{t\to\infty} I(t) = 0.$$

this would mean that everyone gets ill.

(not necessarily easy to answer).

Luckily this model is simple enough that it can be solved exactly!

Namely, $\frac{dI}{dS} = \frac{\dot{I}}{\dot{S}} = -1 + \frac{\nu}{\beta S}$

$$\Rightarrow I(S) = -S + \frac{\nu}{\beta}\log S + \text{constant}$$

$$\Rightarrow \frac{d}{dt}\underbrace{\left(I + S - \frac{\nu}{\beta}\log S\right)}_{\text{"conserved quantity"}} = 0$$

quantity is conserved along solution curves.

(check)

$\Rightarrow$ $\exists!$ soln curve connection each equilibrium inside

$\nu/\beta < S < \infty$ (and $I = 0$)

to an eq. with $0 < S < \frac{\nu}{\beta}$ (and $I = 0$)

$\Rightarrow$ typically $\lim_{t\to\infty} S(t) \neq 0$.

new problem sheet (last one) in the back

SIRS model

new!

* assumption: recovered people may become susceptible again (rate is prop to recovered pop.)

$$\begin{cases} \frac{d}{dt} R = \nu I - \underset{\substack{\uparrow \\ \text{rate of } R \to S}}{\mu R} & \mu > 0. \\ \frac{d}{dt} S = -\beta S I + \mu R \\ \frac{d}{dt} I = \beta S I - \nu I \end{cases}$$

still $\frac{d}{dt}(R + S + I) = 0$. let $\tau = S + I + R$ total pop.

eliminate R by $R = \tau - S - I$

$$\Rightarrow \begin{cases} \frac{d}{dt} S = -\beta S I + \mu(\tau - S - I) \\ \frac{d}{dt} I = \beta S I - \nu I \end{cases}$$

closed system (indep of R)

(and $\dot{R} = \dots$) (recall SIR model when $\mu = 0$)

First observation: no longer a line of equilibria at $I = 0$

Equilibria: $(\tau, 0)$ (no disease present)

$$(S^*, I^*) = \left(\frac{\nu}{\beta}, \frac{\mu(\tau - \frac{\nu}{\beta})}{\nu + \mu} \right)$$ (check)

$\Rightarrow$ (permanent equilibrium between susceptible, infected and recovered population, all nonzero)

Note: (S^*, I^*) makes sense only if both entries are positive

in particular if $\tau > \frac{\nu}{\beta}$ (threshold population $\frac{\nu}{\beta}$)

next thing to look for is stability properties of equilibria:

Jacobian at eq. (S, I) is equal to

$$\begin{pmatrix} -\beta I - \mu & -\beta S - \mu \\ \beta I & \beta S - \nu \end{pmatrix}$$

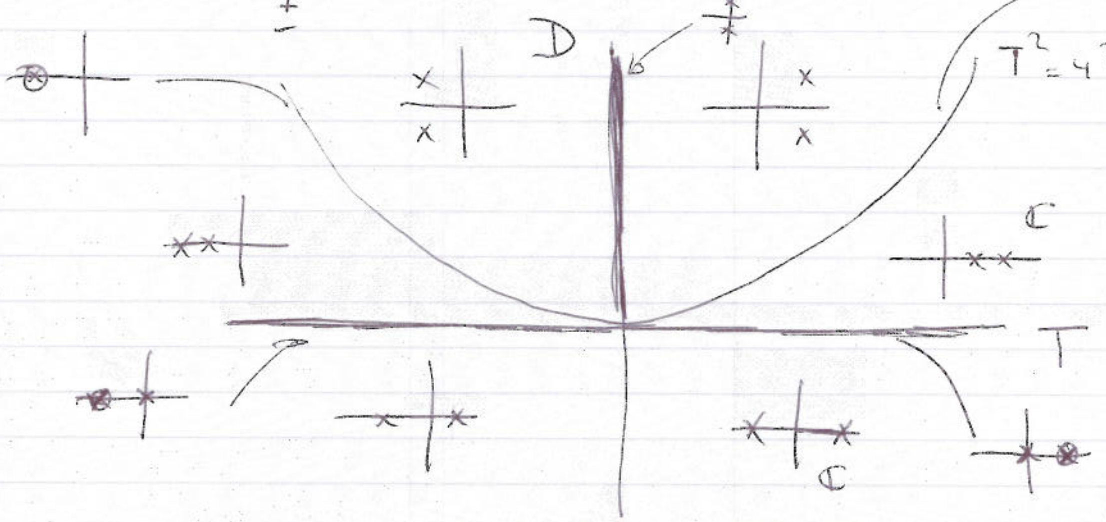
if $(S, I) = (\tau, 0)$ the eigenvalues are $\underset{<0}{\underset{\uparrow}{-\mu}}$ and $\underset{\substack{<0 \text{ below thr.h.} \\ >0 \text{ above thr.h.}}}{\underset{\uparrow}{\beta\tau - \nu}}$

It turns out (short calculation) that the stability of (S^*, I^*) above threshold is <u>asymptotically stable</u> (eigenvalues have negative real part)

<u>digression:</u> how to determine the nature of eigenvalues of a 2×2 matrix A?

<u>observation:</u> eigenvalues are a fn of $\underset{\substack{\| \\ T}}{\mathrm{Tr}\,A}$ and $\underset{\substack{\| \\ D}}{\mathrm{Det}\,A}$

then $\lambda_{\pm} = \frac{1}{2}\left(T \pm \sqrt{T^2 - 4D}\right)$



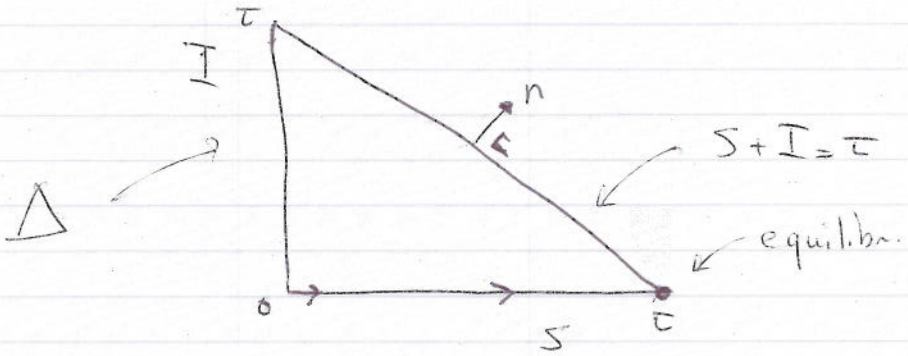
eigenval conf. (HSD chap4)

<u>important lines:</u> $D = 0$, $T = 0$, $T^2 - 4D = 0$

back to the SIRS model:

the region of interest in S-I plane is bounded by

$S = 0$, $I = 0$ and $S + I = \tau$



· note as $\tau < \frac{\nu}{\beta}$ then $(\tau, 0)$ is only equilibrium

· $I = 0$ is flow-invariant subspace: $I = 0 \Rightarrow \dot{I} = 0$

· next question: what is the flow at the boundary $S = 0$ and $S + I = \tau$

at $S = 0$: $\dot{I} = -\nu I \leq 0$

$\dot{S} = \mu(\tau - S - I)$ on relevant domain ≥ 0

like to argue that the flow is inward from the lhs boundary. vector $(\dot{S}, \dot{I})$ on boundary $S = 0$ with $0 \leq I < \tau$ points inside

As a consequence, no solution curve starting inside Δ can escape Δ through lhs boundary
(good! starting from sensible initial condition we cannot end up with negative S!)

Of course, for the model to make sense we also should not be able to escape along $S+I=\tau$ boundary.

<u>How to show this?</u> again we must make sure that the vector field on $S+I=\tau$ points "inward" wrt Δ.

how to show this with a calculation?

Consider outward normal vector to line-segment, here $n = \begin{pmatrix}1\\1\end{pmatrix}$

take inner product ("dot" product) with vector field on pt on this line segment.
* if 0 then vf points in dir of line seg.
* if >0 the vf " outwards
* if <0 the vf points inwards

on line $\tau = S+I$ we have $S = \tau - I$

$$\begin{pmatrix}\dot{S}\\ \dot{I}\end{pmatrix} = \begin{pmatrix}-\beta S I\\ \beta S I - \nu I\end{pmatrix} = \begin{pmatrix}-\beta(\tau - I)I\\ \beta(\tau-I)I - \nu I\end{pmatrix}$$

$$\begin{pmatrix}\dot{S}\\ \dot{I}\end{pmatrix}\cdot\begin{pmatrix}1\\1\end{pmatrix} = \{-\nu I \qquad \nu>0 \quad \forall\, 0 \le I \le \tau$$

indeed inner product is negative for all $I \neq 0$.

We just proved:

<u>Proposition</u>: Δ is positive flow-invariant.

$\Longrightarrow \forall x \in \Delta \quad \omega(x) \subset \Delta$

~~∃ ... then~~

$\Rightarrow$ (1) below threshold $\exists!$ equilibrium $(\tau, 0)$

claim: $\omega(x) = \{(\tau,0)\} \quad \forall x \in \Delta$

if $\omega(x)$ contains eq $\Rightarrow \omega(x) = \{(\tau,0)\}$

if not, then $\omega(x)$ is per soln $\subset \Delta$ but this p.s must encircle an equilibrium pt $\lightning$ (no such eq. pt).

(2) if above threshold $\Rightarrow \exists$ asympt stable eq (S^*, I^*) and $(\tau, 0)$ saddle pt

Last lecture: tuesday 24/3.
last problem class: wed 24/3 (but I will not be there ☺? ☹

⊙ Keep an eye on website! ("how to study for exam")

Predator-prey systems:

prey population x

predator pop y

if no predators $\Rightarrow$ x grows with constant rate $\dot{x} = ax$ $a>0$

with predators we have additive decay, proportional to pred pop.

$$\dot{x} = ax - bxy \qquad b>0.$$

pred. pop: in absence of prey y declines with constant rate.

$$\dot{y} = -cy \qquad c>0$$

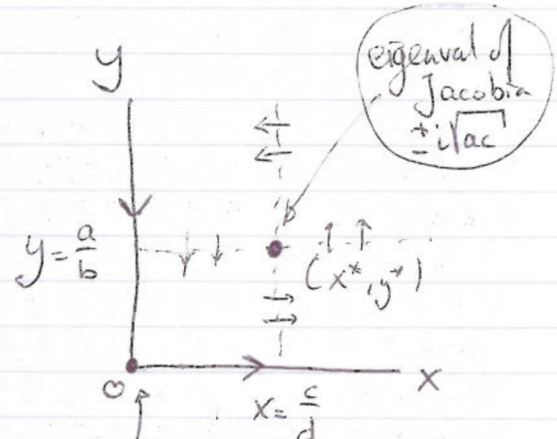
in presence of prey we have growth prop. to prey pop.

$$\dot{y} = -cy + dxy \qquad d>0.$$

$\Rightarrow$ system

$$\begin{cases} \dot{x} = x(a-by) \\ \dot{y} = y(-c+dx) \end{cases} \qquad x, y \geq 0.$$

inconclusive!



equilibria: (nullclines).

$\dot{x} = 0$ iff $x = 0$ $\vee$ $y = \frac{a}{b}$

$\dot{y} = 0$ iff $y = 0$ $\vee$ $x = \frac{c}{d}$

x-axis, y-axis flow invariant

saddle
(hyp with nontrivial stable and unstable manifolds)

Usually one would be stuck here, not knowing about Lyap or asympt stability of $(x^*, y^*) = (\frac{c}{d}, \frac{a}{b})$, with possibility of limit cycles encircling this equilibrium.

19/3
II

It turns out that this system admits a Lyapunov function (lucky!).
Can be derived using "trial" Lyap fn of the form $L(x,y) = F(x) + G(y)$
Then if insisting $\frac{d}{dt} L = 0 \Rightarrow$ (see HSD notes)

$$L(x,y) = dx - c\log x + by - a\log y$$

one can check that $\frac{d}{dt} L(x,y) = 0$.

moreover (x^*, y^*) is point where L is minimal.

$\Rightarrow$ $V(x,y) = L(x,y) - L(x^*,y^*)$ has all prop of Lyap fn.

$$\frac{d}{dt} V(x,y) = 0.$$

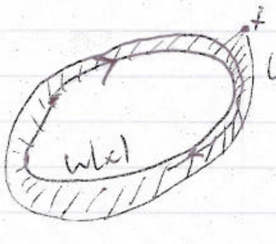
V is conserved! but V is not constant on any open set.

Now we observe the following:

if a planar flow has a conserved quantity V and ~~L~~ V is not constant on any open set then the flow cannot have an asymp stable equilibrium point, nor any solution that "converge to a periodic solution" (i.e. there is no x such that $\omega(x)$ is a periodic soln and $x \notin \omega(x)$).

Proof: (by picture)

U nbh converging to eq.
$\Rightarrow V$ is constant on U
$\forall x \in U \quad V(x) = V(x_0)$



U open nbh.
$\forall x \in U \quad V(x) = V(y)$
$y \in \omega(x)$

$\Rightarrow$ (x^*, y^*) cannot be asympt stable for (forward and backward) flow.

$\Rightarrow$ (x^*, y^*) is Lyapunov stable.

Still open question: are there any periodic solns? Answer: yes!

Proposition: every initial condition that is not equal to one of the two equilibria, lies on a periodic solution encircling (x^*, y^*)

(periodic solns lie exactly on level sets of V, i.e. V=constant)

Proof: by Poincaré map or flow confined to level sets of V.

A problem with this model is that it contains the rather unrealistic scenario that the prey population explodes (unbounded growth) (because planet has finite area).

We will modify it in the following way.

$$\dot{x} = \underset{\substack{\uparrow \\ \text{growth}}}{ax} - \underset{\substack{\uparrow \\ \text{overcrowding damping}}}{\lambda x^2} = x(a - \lambda x) \qquad \lambda > 0.$$

if $x > \frac{a}{\lambda}$ then pop of prey decreases by itself.

Without predators, pop of prey converges to $x = \frac{a}{\lambda}$

(in book HSD, one also introduces $\dot{y} = -cy \underset{\substack{\uparrow \\ \mu>0}}{-\mu y^2}$

avoiding overcrowding of predators)

As before, the parameters a, b, c, d as well as λ and μ are all positive. When $y = 0$, we have the logistic equation $x' = x(a - \lambda x)$, which yields equilibria at the origin and at $(a/\lambda, 0)$. As we saw in Chapter 1, all nonzero solutions on the x-axis tend to a/λ.

When $x = 0$, the equation for y is $y' = -cy - \mu y^2$. Since $y' < 0$ when $y > 0$, it follows that all solutions on this axis tend to the origin. Thus we confine attention to the upper-right quadrant $\mathcal{Q}$ where $x, y > 0$.

The nullclines are given by the x- and y-axes, together with the lines

$$\begin{aligned} L: &\quad a - by - \lambda x = 0 \\ M: &\quad -c + dx - \mu y = 0. \end{aligned}$$

Along the lines L and M, we have $x' = 0$ and $y' = 0$, respectively. There are two possibilities, according to whether these lines intersect in $\mathcal{Q}$ or not.

We first consider the case where the two lines do not meet in $\mathcal{Q}$. In this case we have the nullcline configuration depicted in Figure 11.6. All solutions to the right of M head upward and to the left until they meet M; between the lines L and M solutions now head downward and to the left. Thus they either meet L or tend directly to the equilibrium point at $(a/\lambda, 0)$. If solutions cross L, they then head right and downward, but they cannot cross L again. Thus they too tend to $(a/\lambda, 0)$. Thus all solutions in $\mathcal{Q}$ tend to this equilibrium point. We conclude that, in this case, the predator population becomes extinct and the prey population approaches its limiting value of a/λ.

We may interpret the behavior of solutions near the nullclines as follows. Since both x' and y' are never both positive, it is impossible for both prey

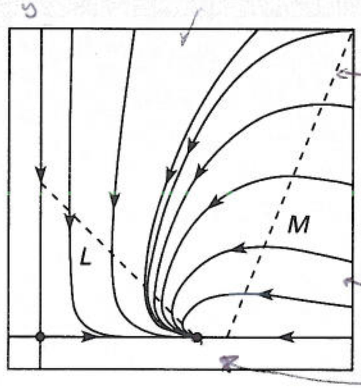


Figure 11.6 The nullclines and phase portrait for a predator/prey system with limited growth when the nullclines do not meet in $\mathcal{Q}$.

flow invariant box

∄ no periodic soln!

hence converge to

predators become extinct

and predators to increase at the same time. If the prey population is above its limiting value, it must decrease. After a while the lack of prey causes the predator population to begin to decrease (when the solution crosses M). After that point the prey population can never increase past a/λ, and so the predator population continues to decrease. If the solution crosses L, the prey population increases again (but not past a/λ), while the predators continue to die off. In the limit the predators disappear and the prey population stabilizes at a/λ.

Suppose now that L and M cross at a point $Z = (x_0, y_0)$ in the quadrant $\mathcal{Q}$; of course, Z is an equilibrium. The linearization of the vector field at Z is

$$X' = \begin{pmatrix} -\lambda x_0 & -bx_0 \\ dy_0 & -\mu y_0 \end{pmatrix} X.$$

The characteristic polynomial has trace given by $-\lambda x_0 - \mu y_0 < 0$ and determinant $(bd + \lambda\mu)x_0 y_0 > 0$. From the trace-determinant plane of Chapter 4, we see that Z has eigenvalues that are either both negative or both complex with negative real parts. Hence Z is asymptotically stable.

Note that, in addition to the equilibria at Z and $(0, 0)$, there is still an equilibrium at $(a/\lambda, 0)$. Linearization shows that this equilibrium is a saddle; its stable curve lies on the x-axis. See Figure 11.7.

It is not easy to determine the basin of Z, nor do we know whether there are any limit cycles. Nevertheless we can obtain some information. The line L meets the x-axis at $(a/\lambda, 0)$ and the y-axis at $(0, a/b)$. Let Γ be a rectangle whose corners are $(0, 0)$, $(p, 0)$, $(0, q)$, and (p, q) with $p > a/\lambda$, $q > a/b$, and the point (p, q) lying in M. Every solution at a boundary point of Γ either

possible scenario

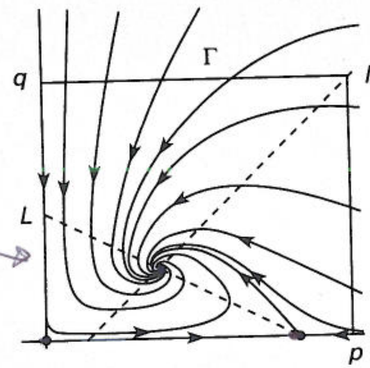


Figure 11.7 The nullclines and phase portrait for a predator/prey system with limited growth when the nullclines do meet in $\mathcal{Q}$.

~~Not~~ sure that pt converge to ω-limit set but could be eq. or = periodic soln surrounding the eq.