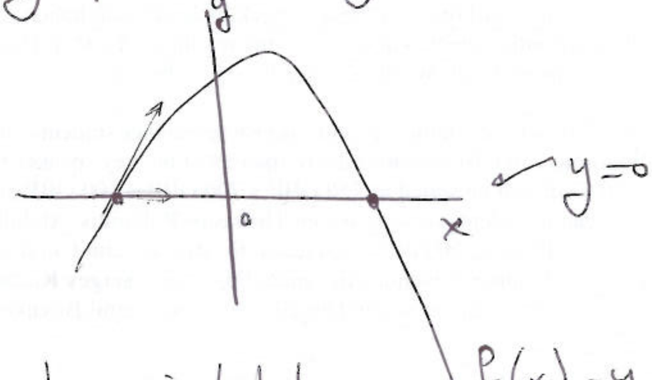
$m=1 \qquad x, y \in \mathbb{R}$

$f(x)=0$ we view this soln set as the intersection of $y=f(x)$ and $y=0$ in the x,y-plane.



from picture we have isolated equilibria if $f(x)=y$ and $y=0$ intersect transversely.

<u>condition for transversality:</u>

$y=0$ has tangent space $\left\langle \begin{pmatrix}1\\0\end{pmatrix} \right\rangle$

the curve $(x, f(x))$ has tangent space in x_0

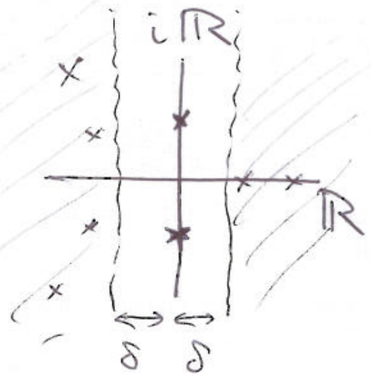
$$\left\langle \begin{pmatrix}1\\f'(x_0)\end{pmatrix} \right\rangle$$

$\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}1\\f'(x_0)\end{pmatrix}$ are linearly indeps if $f'(x_0)\neq 0$.

(from <u>exercises</u>) we know that the transverse intersection of two curves in $\mathbb{R}^2$ has dimension $-(2-1-1)=0 \Rightarrow$ i.e. <u>isolated point</u>.

$\uparrow$ $\mathbb{R}^2$ $\quad\uparrow$ curve $\quad\uparrow$ curve

(a)



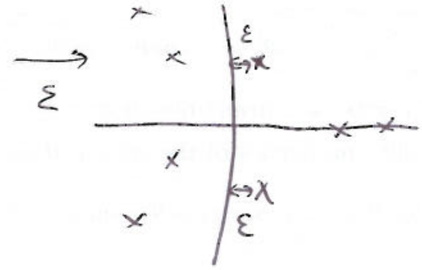
eigenvals of $A \in gl(m, \mathbb{R})$

What are evals of $A + \varepsilon I$?

if v eigenvector of A with eval λ : $Av = \lambda v$

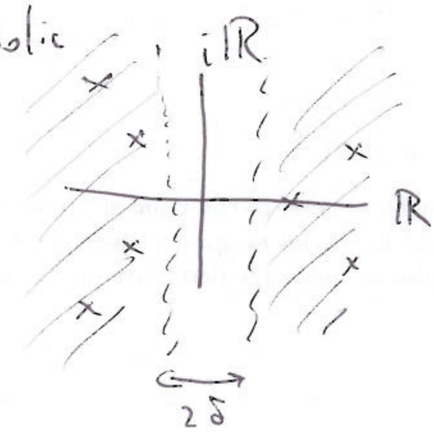
then $(A + \varepsilon I) v = Av + \varepsilon v = \lambda v + \varepsilon v = (\lambda + \varepsilon) v$

So A and $A + \varepsilon I$ have the same eigenvectors and the evals of the latter are the ones of A shifted by ε.



if $|\varepsilon| < \delta$ $\varepsilon \neq 0$
then $A + \varepsilon I$ has
<u>no</u> evals on $i\mathbb{R}$.

(b) A hyperbolic



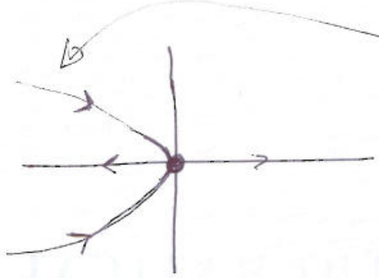
What are eigenvals of $A + B$ when $|B| \leq \delta$.

Claim $A + B$ also hyperbolic.

EXERCISE

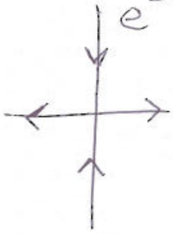
$$|B| = \sup_{|x| \neq 0} \frac{|Bx|}{|x|} = \sup_{|x|=1} |Bx| \approx \text{largest absolute value of eigenvalue of } B$$

The desired result then follows by (Real part of) application of the triangle inequality or $|Ax| \geq \delta |x|$ and $|Bx| \leq \delta |x|$ $\Rightarrow |(A+B)x| > 0$.



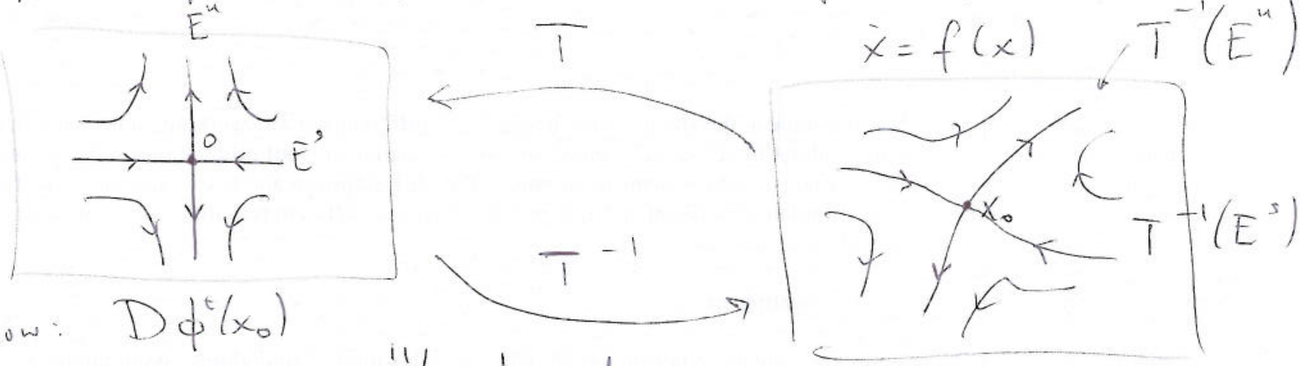
?

cp. to linear flow



exp rate of contraction and expansion

$\dot{x} = \cancel{D\phi^t(x_0)} \approx Df(x_0)x$ $\qquad$ flow: ϕ^t



flow: $D\phi^t(x_0)$

"topological conjugacy" $\qquad T^{-1}(0) = x_0$

$$T \circ \phi^t \circ T^{-1} = D\phi^t(x_0)$$

composition

$$\Rightarrow \quad \phi^t = T^{-1} \circ D\phi^t(x_0) \circ T$$

if $x(t)$ soln of $\dot{x} = Df(x_0)x$

then $T^{-1}(x(t))$ soln of $\dot{x} = f(x)$

In particular: $\exists!$ curve $T^{-1}(E^s)$ on which all initial conditions converge to x_0 as $t \to \infty$

$\exists!$ curve $T^{-1}(E^u)$ .. .. $t \to -\infty$

I

19/2 New problem sheet in the back! (Solns to sheet 4 on web later today.)

Continuation

Consider $\frac{dx}{dt} = f(x,\lambda) \quad x \in \mathbb{R}^m, \lambda \in \mathbb{R} \quad (\text{or } \lambda \in \mathbb{R}^p)$

suppose that f has an equilibrium x_0 at $\lambda = 0$
i.e. $f(x_0, 0) = 0$

Question: what happens if λ is changed (slightly)?
(a) * is there still an equilibrium?
(b) * if so, is the dynamics (flow) near the equilibrium changed, and if so in what respect?

(a) first note that x_0 is typically isolated

↑ $D_1 f(x_0, 0)$ is invertible

↑ derivative wrt x

if this is the case $\Rightarrow$ if λ is small enough locally there remains to be a single isolated equilibrium.

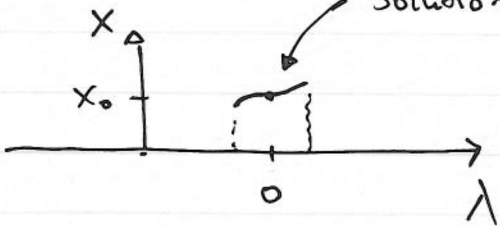
why? because of the IFT!

if $f(x_0,0)=0$ and $D_1 f(x_0,0)$ is invertible
$\Rightarrow \exists!\ x(\lambda)$ with $x(0) = x_0$ such that

$$f(x(\lambda), \lambda) = 0 \quad \text{if } |\lambda| \text{ suff. small}$$

II 19/2

consider schematic representation

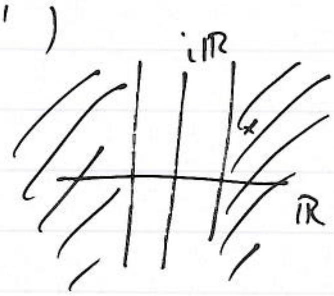


this is called "<u>continuation</u>" of the equilibrium
(as a fn of parameter(s))

What about the local flow?

* typically the isolated equilibrium is hyperbolic.
  this property is <u>persistent</u> under small perturbation

because $D_1 f$ is assumed to be continuous
($f \in C^1$)



Hence, if x_0 is hyperbolic
then for λ suff small
we also find that
$x(\lambda)$ is also hyperbolic

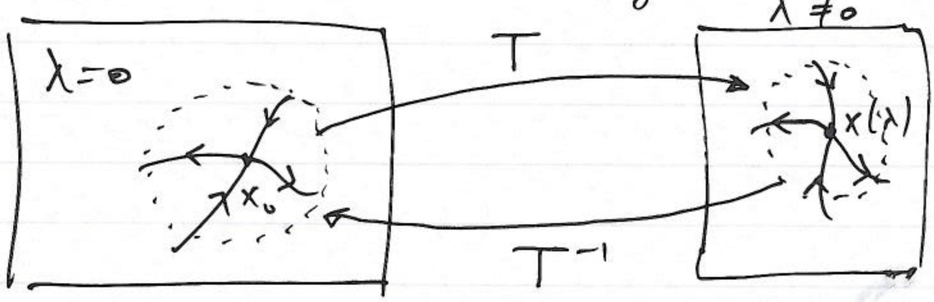
<u>Question</u>: what if x_0 (at $\lambda=0$) is not hyperbolic?

i.e. $D_1 f(x_0, 0)$ has eval on $i\mathbb{R}$

<u>first remark</u>: if still $D_1 f(x_0, 0)$ is invertible
(i.e. no zero eigenvalue)
then $\exists!$ $x(\lambda)$ curve of isolated equilibria
but not necessarily
sure of local flow.

III 19/2

Going back quickly to the family of hyperbolic equilibria, we know not only that $x(\lambda)$ is hyperbolic $|\lambda|$ suff small, but also that $\dim W^s(x(\lambda))$ and $\dim W^u(x(\lambda))$ are unchanged.



$\exists$ coordinate transf T such that the soln curves near x_0 at $\lambda = 0$ match exactly the soln curves near $x(\lambda)$ at λ (by application of Hartman-Grobman thm)

~~Note: $\exists$ can not match the time parametrization~~

Even though the flows are topologically conjugate, there are of course differences:

- evals of $D_1 f(x_0, 0)$ are different from
  " " $D_1 f(x(\lambda), \lambda)$

  $\Rightarrow$ different rates of contraction and expansion on W^s and W^u

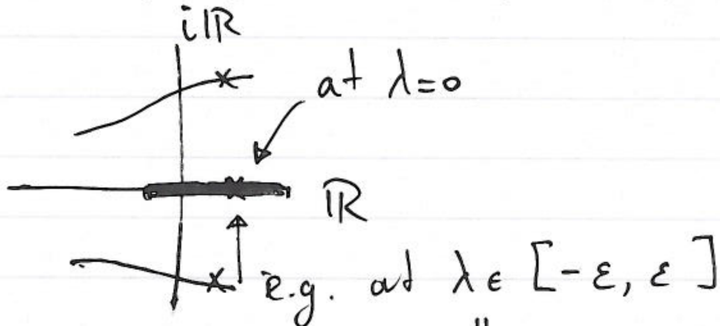
IV 19/2

if <u>non-hyperbolic</u>: $\boxed{\begin{array}{c}(a)\\(b)\end{array}}$ (a) zero eigenvalue of $D_1 f(x_0, 0)$

(b) pair of c.c. eigenvalue $\pm ia$ of $D_1 f(x_0, 0)$

if $\lambda \in \mathbb{R}$ then "typically" we would only expect to see non-hyperbolic equilibria with property (a) or (b).

in the 1-parameter family of vector fields $f(x, \lambda)$



$\Downarrow$ curves of eigenvalues.

Consider eigenvalues of $D_1 f(x(\lambda), \lambda)$ as fn of λ.

If any such curve intersects the $i\mathbb{R}$ transversely then under additional small perturbation of the vector field, such an intersection will persist.

First, consider (a): $D_1 f(x_0, 0)$ has zero eigenvalue.

<u>Example</u>: $\cancel{f(x,\lambda) = a + \lambda x + bx^2 + \lambda \quad f: \mathbb{R} \to \mathbb{R}}$ $\cancel{a, b \in \mathbb{R}}$.

$$f(x, \lambda) = \lambda + a x^2 \qquad \dot{x} = f(x) \quad x \in \mathbb{R}.$$

$$a \neq 0.$$

$$f(0,0) = 0 \qquad D_1 f(0,0) = 0$$

$f(x,\lambda) = 0 \iff x^2 = -\dfrac{\lambda}{a}$ if $\dfrac{\lambda}{a} > 0 \Rightarrow$ no <u>soln</u>

if $\dfrac{\lambda}{a} < 0 \Rightarrow$ 2 <u>solns</u>.

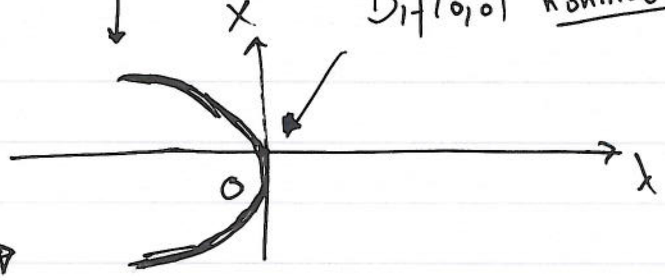
V 19/2

$x^2 = -\frac{\lambda}{a}$

$D_1 f(0,0)$ noninvertible.

$a > 0$

"fold" singularity



this is an example of a bifurcation. ("change of dynamics")

$\lambda < 0$ 2 equilibria

$\lambda > 0$ no equilibri.