$\phi^t : \mathbb{R} \to \mathbb{R}$

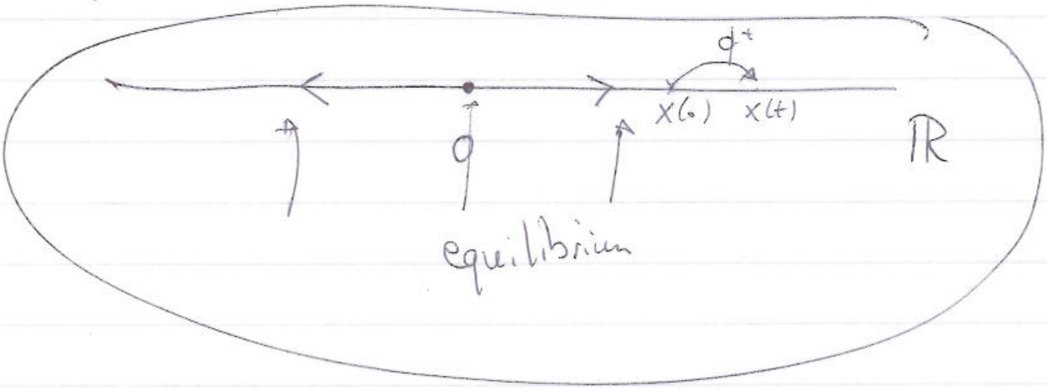
$\phi^{t,\tau}(x) = x \cdot \exp(t-\tau)$

$\phi^t(x) = x \cdot \exp(t)$

$$\boxed{\frac{dx}{dt} = x \qquad x_\tau = x(\tau)}$$

analytical

$$x(t) = x_\tau \exp(t-\tau)$$



geometric

<u>flow</u>: $\phi^t : \mathbb{R} \to \mathbb{R}$

examples: $\phi^t(0) = 0 \quad \forall t$

if $x > 0 \quad \phi^t(x) > x \quad \forall t > 0$

if $x < 0 \quad \phi^t(x) < x \quad \forall t > 0$

$\phi^t(x(0)) = x(t)$

<u>if f is smooth then also ϕ^t is smooth</u>

v eigenvect of A $\quad Av = \mu v$

then we can restrict the ODE $\frac{dx}{dt} = Ax$

to the eigenspace $\langle v \rangle$ (linear v. space gen. by v)

and we have $\frac{d}{dt}(av) = A(av) = a\mu v$

$a \in \mathbb{R}$

$$\Rightarrow \frac{d}{dt} a = \mu a \quad \Rightarrow a(t) = a(0)e^{\mu t}$$

So the eigenspaces of A are flow-invariant ~~eigen~~ subspaces of the flow ϕ^t associated to $\frac{dx}{dt} = Ax$

i.e if a solution $x(t)$ has one pt in an eigenspace of A, then $x(t)$ lies <u>entirely</u> in this eigenspace.

$$\frac{dx}{dt} = f(x)$$ ODE $x \in \mathbb{R}^m$

$f: \mathbb{R}^m \to \mathbb{R}^m$ "vector field"

base space

vector space

$$\frac{d\vec{x}}{dt} = A\vec{x} \qquad A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \qquad \vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{cases} \dfrac{dx}{dt} = \lambda_1 x & \qquad x(t) = e^{\lambda_1 t} x(0) \\ \dfrac{dy}{dt} = \lambda_2 y & \qquad y(t) = e^{\lambda_2 t} y(0) \end{cases}$$

$$\begin{pmatrix} x \\ y \end{pmatrix}(t) = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix}$$

$$\phi^t = \exp(At)$$

$$\dot{\vec{x}} = A\vec{x} \quad \begin{cases} \dfrac{dx}{dt} = -\beta y \\ \dfrac{dy}{dt} = \beta x \end{cases} \quad \Rightarrow \quad \frac{d^2x}{dt^2} = -\beta \frac{dy}{dt} = -\beta^2 x$$

(harmonic oscillator!)

solutn $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \cos(\beta t) & -\sin(\beta t) \\ \sin(\beta t) & \cos(\beta t) \end{pmatrix} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix}$

should be familiar.

flow map $\phi^t = \exp(At) : \mathbb{R}^2 \to \mathbb{R}^2$

$A = \begin{pmatrix} 1 & 2 \\ -1 & 2 \end{pmatrix} \qquad A^k = ?$

Questn: how to understand $\exp(A)$?

First thing of importance: eigenvalues of A.

one scheme (say for $A \in gl(2, \mathbb{R})$):

choose convenient coordinates (eg so that A is in Jordan form)

Let Jordan form of A be L, such that $A = PLP^{-1}$ for some $P \in GL(2, \mathbb{R})$

general linear group (invertible matrices)

$$\exp(A) = \exp(PLP^{-1}) = \sum_{k=0}^{\infty} \frac{(PLP^{-1})^k}{k!} = P\left(\sum_{k=0}^{\infty} \frac{L^k}{k!}\right)P^{-1} = P\exp(L)P^{-1}$$

this is one possible way to compute $\exp(A)$.