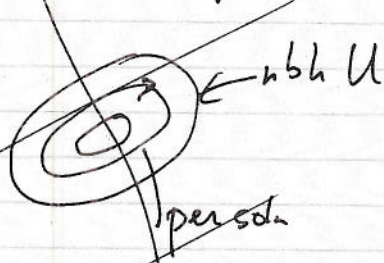
<u>important observation</u>: if $\underset{S}{\hat{z}} \in \omega(x)$

then $\exists$ nbh of z in S such that z is the unique intersection of $\omega(x)$ with S (as in yesterday's lecture!)

<u>Some corollaries</u>:

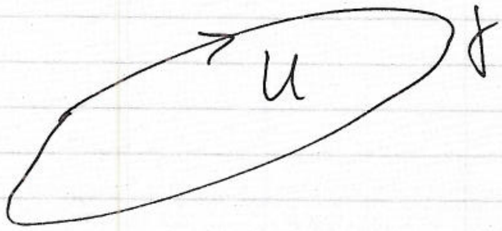
(1) $\exists$ open ~~nbh~~ set U of any isolated periodic solution γ



~~such that $\omega(x) = \gamma \quad \forall x \in U$~~

~~$\exists$ open set U such that if γ is a periodic soln <u>and</u> ω-limit set, then $\gamma = \omega(x) \quad \forall x \in U$~~

(1) Let γ be a periodic solution an U be the area enclosed by γ.

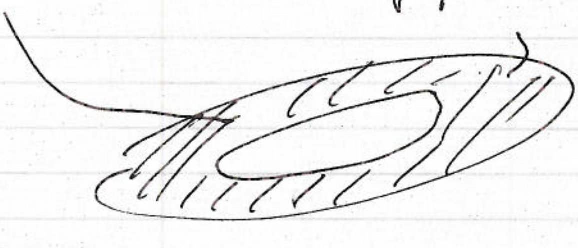


$\gamma \cup U$ is compact forward and backward flow invariant $\Rightarrow$ contains ω-limit set and also α-limit set

$\Rightarrow U$ contains a periodic soln <u>or</u> an equilibrium pt

$U \cup \gamma$ is closed // ~~But~~ $\Rightarrow \gamma$ could serve as ω-limit set for pts in U, but then it could <u>not</u> serve as α-limit set.

Suppose γ is ω-limit set for some pts in U then by monotonicity of the return map, it is the ω-limit set of an entire annulus on the "boundary of U"



$\Rightarrow$ U contains an α-limit set

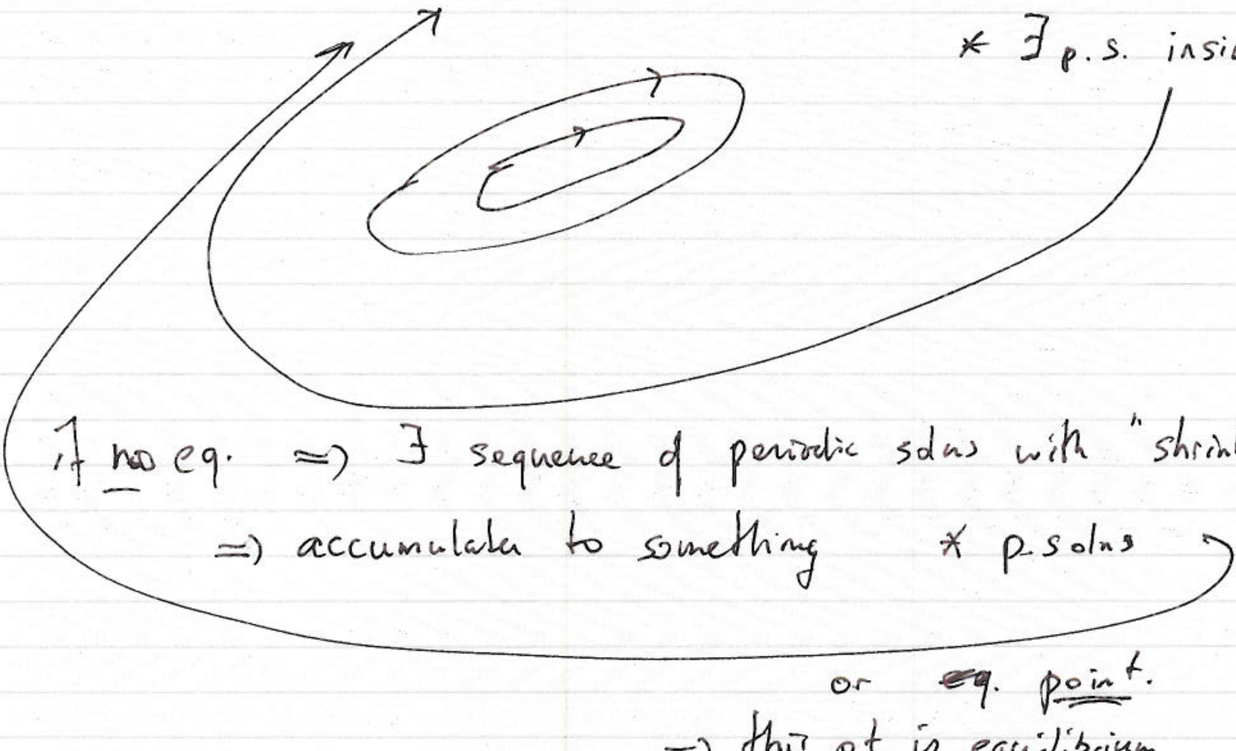
(recall α-limit set is ω-limit set of backward flow)

$\Rightarrow$ PB U contains an equilibrium or periodic soln.

(2) U (as above) contains an equilibrium pt.

<u>proof</u>: struct as above $\Longrightarrow$ * $\exists$ eq. <u>done</u>

* $\exists$ p.s. inside U



if <u>no</u> eq. $\Rightarrow$ $\exists$ sequence of periodic solns with "shrinking radius"

$\Rightarrow$ accumulate to something * p.solns

or ~~eq.~~ <u>point</u>.

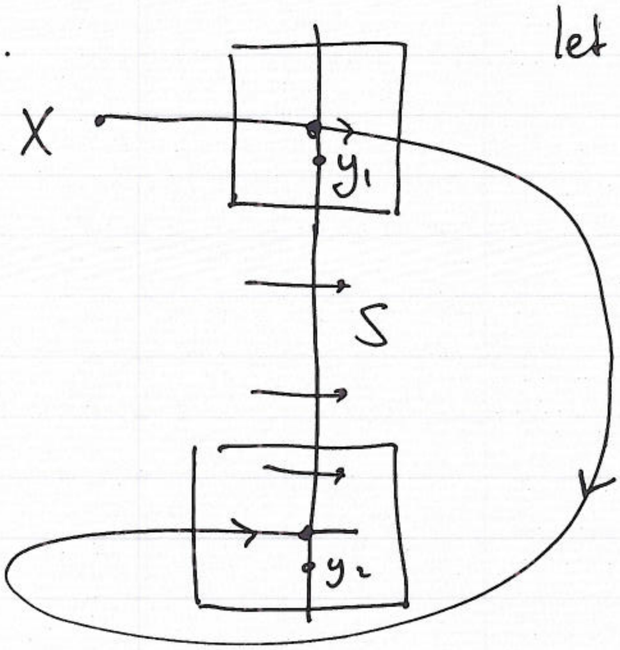
$\Rightarrow$ this pt is equilibrium $\lightning$

note: this observation is very special (to $\mathbb{R}^2$)

(3) as a consequence of (2) we have the following:

Proposition: suppose $y \in \omega(x)$, then $\{\phi^t(y)\}_{t\in\mathbb{R}}$ crosses any local section S in no more than one point.

let $y_1, y_2 \in \{\phi^t(y)\}_{t>0}$



since $y \in \omega(x) \Rightarrow y_1, y_2 \in \omega(x)$

suppose $y_2 = \phi^{\tau}(y_1)$ $\tau > 0$.

~~so~~ ~~If~~ By monotonicity of the return map, $\{\phi^t(x)\}_{t>0}$ cannot accumulate to both y_1 and y_2

q.e.d. every orbit inside an ω-limit set ~~intersection~~ intersects any section at most once.

$\Rightarrow$ Huge limitations to ω-limit sets.

11/3

I

Poincaré–Bendixson theorem

ω-limit sets on the plane for aut. ODE are:

* equilibria
* periodic solns
* network of equilibria with connecting orbits.

∃ equilibria inside ω-limit set

special case:

Thm: let $\Omega \subset \mathbb{R}^2$ be nonempty, closed and bounded (compact) ω-limit set of an aut. ODE in $\mathbb{R}^2$ that contains no equilibrium,

then Ω is a periodic solutions.

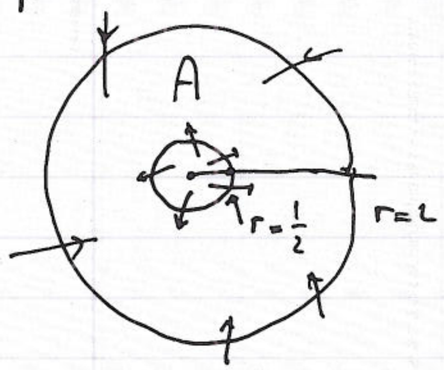
Example:
$$\begin{cases} \dot\theta = 1 + \frac{1}{10}\cos^2(\theta) \\ \dot r = r(1-r) + \frac{1}{8}\sin(\theta) \end{cases}$$

if we add these terms we can no longer find explicit solutions, but the same conclusion still holds

if $r \gg 1$ then flow is towards origin ($\dot r < 0$)

if $r \simeq 0$ (<1) th " " away from origin ($\dot r > 0$)

$r=0$ eq. pt.



flow is transverse to boundary of A and in-flowing

$\Rightarrow$ A is forward-invariant set of the flow.

$\Rightarrow$ A contains ω-limit set

we observe that A does not contain an equilibrium.

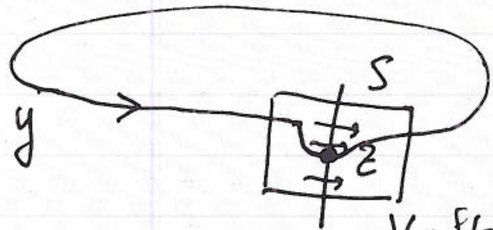
$\Rightarrow$ PB: A contains a periodic solution (which is an ω-limit set)

(note: it is not clear à priori if there is only one per. soln!)

proof: let $\Omega = \omega(x)$ for some $x \in \mathbb{R}^2$
be closed and bounded.
let $y \in \omega(x)$ then $\omega(y) \subset \omega(x)$

now let $z \in \omega(y)$ such that z is not an equilibrium

let S be local section transverse to the flow at z



since $\forall$ sequence $\{t_n\}_{n\in\mathbb{N}}$ $\lim_{n\to\infty} t_n = \infty$ such that

$\lim_{n\to\infty} \phi^{t_n}(y) = z$ the forward orbit of y

$\{\phi^t(y)\}_{t\geq 0}$ must enter and exit V and ∞ number of times.
But $y \in \omega(x) \Rightarrow \{\phi^t(y)\}_{t\geq 0} \subset \omega(x)$

$\Rightarrow$ (yesterday's lecture) $\{\phi^t(y)\}_{t\geq 0}$ intersects S at most in one point.

$\Rightarrow \exists\, t, s > 0 \quad t \neq s$ such that $\phi^t(y) = \phi^s(y) \in S$

$\Rightarrow \phi^{t-s}(y) = y \Rightarrow y$ lies on a periodic soln

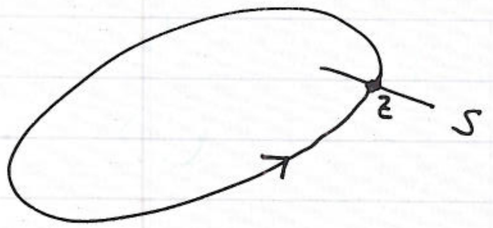
since $\{\phi^t(y)\}_{t\geq 0}$ accumulates to z, z also must lie on this periodic solution.

11/3 III

next: if $\omega(x)$ contains a periodic solution then $\omega(x)$ is a periodic solution.

Consider the periodic solution inside $\omega(x)$

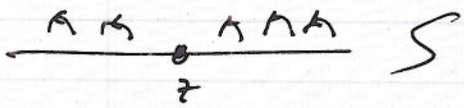
let z be a pt on this periodic soln



Now we will use the monotonicity of the return map $P : S \to S$:

either, on each side of S, pts go towards z under the map P, or pts go away from z

in particular pts cannot go towards z and then decide at some pt to go away



$\Rightarrow$ $\omega(x)$ containing periodic soln must be equal to the periodic soln, since if $\exists$ sequence $\{t_k\}_{k \in \mathbb{N}}$

s.t. $\quad \phi^{t_k}(x) \xrightarrow{k \to \infty}$ periodic soln

$\phi^{t_k}(x)$ cannot converge to anything else.

Note here that if y is close to z, $\{\phi^t(y)\}_{t \in \mathbb{R}}$ is also close to $\{\phi^t(z)\}_{t \in \mathbb{R}}$ for some finite time-interval.

(by continuity of the flow).