M2AA1 Differential Equations

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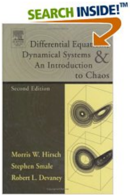
Basic facts

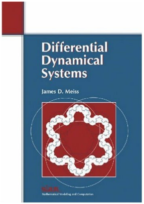
- There are three one-hour lectures per week + one example class.
- There will be one problem sheet per week.
- There will be two progress tests (each counting for about 5% of the total exam mark), dates TBA.

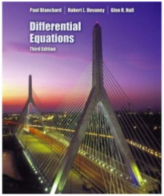
Support

- Course notes will be made available as the course progresses (covering all lectures).
- Solutions will be made available for all problem sheets (with some delay).
- Towards the end of term I will provide detailed instructions on the form of the exam and how to study for it.
- All available information about this course is available from the course website: `http://www.ma.imperial.ac.uk/~mdynamic/M2AA1`
- For any questions, comments, suggestions: blog on the course website or e-mail `jswlamb@ic.ac.uk`

Reading list (recommended but not compulsory)







- Morris Hirsch, Stephen Smale, Robert Devaney. *Differential Equations, Dynamical Systems, and an Introduction to Chaos*, Academic Press, 2003 (34 GBP)
- James D. Meiss. *Differential Dynamical Systems*, SIAM, 2007 (41 GBP)
- Paul Blanchard, Robert L. Devaney, Glen R. Hall. *Differential Equations*, 3rd Edition (includes CD-ROM), 2006 (86 GBP) In library but DO NOT BUY.

What is an ordinary differential equation?

ODE

$$\frac{dx}{dt} = f(x, \lambda, t). \tag{1}$$

with $f : \mathbb{R}^m \times \mathbb{R}^p \times \mathbb{R} \to \mathbb{R}^m$.

- $x \in \mathbb{R}^m$ (phase space)
- $\lambda \in \mathbb{R}^p$ (parameters)
- t is the independent (*time*) variable.
- We assume that f is smooth in all its arguments.
- If f does not explicitly depend on t, the ODE is called *autonomous*. We will mainly consider autonomous ODEs in this course.

General objectives of the study of ODEs

- to understand the set of solutions $x(t, \lambda)$ of (1) for a fixed choice of parameters λ,
- to understand how this set of solutions changes as λ is being varied.

Unfortunately, except in special cases, these objectives remain well beyond (our and everyone's) reach. So we will address mainly more modest objectives.

The main aim of this course

To develop a geometric point of view to the solutions $x(t)$ of (1), and then use this geometric point of view to analyze properties of the solutions of ODEs.

2nd order ODEs

ODEs form the backbone of models for so-called *dynamical systems* that describe the evolution of variables characterizing a system in the course of time.
In mechanics, the equations of motion are often written in the form

$$\frac{d^2x}{dt^2} = g(\frac{dx}{dt}, x, \lambda, t).$$

We call this a *second order* ODE, referring to the highest order derivative of x that appears.
By introducing the auxiliary variable $y = \frac{dx}{dt}$, we can rewrite this differential equation in the form

$$\left\{ \begin{array}{rcl} \frac{dy}{dt} & = & g(y, x, \lambda, t), \\ \frac{dx}{dt} & = & y. \end{array} \right. \quad (2)$$

The equations now have the general (1st order) ODE form that we introduced before.

Mechanics example

Newton's equations of motion for a damped harmonic oscillator with mass m, natural frequency ω and damping coefficient μ are

$$m\ddot{x} = -\omega^2 x - \mu\dot{x}.$$

This can be rewritten as

$$\begin{cases} \frac{dy}{dt} &= \frac{1}{m}(-\omega^2 x - \mu y), \\ \frac{dx}{dt} &= y, \end{cases}$$

so that the ODE takes the form

$$\frac{d}{dt}\begin{pmatrix} y \\ x \end{pmatrix} = f(\begin{pmatrix} y \\ x \end{pmatrix}, m, \omega, \mu) = \begin{pmatrix} \frac{1}{m}(-\omega^2 x - \mu y) \\ y \end{pmatrix}.$$

where $\lambda = (m, \omega, \mu)$ are the parameters.

*k*th order ODEs

In general we say that a differential equation of the form

$$\frac{d^k x}{dt^k} = g(\frac{d^{k-1}x}{dt^{k-1}}, \ldots, \frac{dx}{dt}, x, \lambda, t),$$

is a *k*th order ODE. Along the lines of the above discusssion, it follows that such an ODE can always be written in the form of a first order ODE, by introduction of the auxiliary variables $y_n = \frac{d^{k-n}x}{dt^{k-n}}$ where $n = 1, \ldots, k-1$:

$$\begin{cases} \frac{dy_1}{dt} & = & g(y_1, \ldots, y_{k-1}, x, \lambda, t), \\ \frac{dy_2}{dt} & = & y_1, \\ \vdots & = & \vdots \\ \frac{dy_{k-1}}{dt} & = & y_{k-2}, \\ \frac{dx}{dt} & = & y_{k-1}, \end{cases}$$

Existence and uniqueness of solutions

A very important property of (smooth) ODEs is the *existence and uniqueness of solutions*: if we specify the state of a system at a point x_τ in the phase space (at a given moment in time $t = \tau$), then there is a unique solution $x(t)$ of the ODE so that $x(\tau) = x_\tau$:

- each *state* of the system can be attained,
- and has a unique past and a unique future (for some small time interval, at least)

Existence and uniqueness may seem an obviously desirable property of any model, but there are many differential equations (other than ODEs) that do not have this property.

The development of the analytical tools necessary to prove this result (and many other results) is the main technical objective of this course.

Existence and uniqueness: simple example

Example

Consider the ODE $\frac{dx}{dt} = x$, $x \in \mathbb{R}$. With initial condition $x(\tau) = x_\tau$ this has the unique solution $x(t) = x_\tau \exp(t - \tau)$.
Existence and uniqueness means that through every point x_τ of the phase space $\mathbb{R}$ there exists a unique solution that intersects x_τ at $t = \tau$, namely $x(t) = \exp(t - \tau)x_\tau$.
(Note that in this simple example there are really only three different solutions, which together cover the entire phase space $\mathbb{R}$.)

Flow (for autonomous ODE)

Because of the property of *existence and uniqueness*, instead of thinking of solutions of ODEs as functions $x(t)$ with $x : \mathbb{R} \to \mathbb{R}^m$ satisfying (1), we prefer to view an ODE as inducing a *flow*, i.e. a transformation of the phase space, that represents the *dynamics*:

$$\Phi_\lambda^t : \mathbb{R}^m \to \mathbb{R}^m.$$

$\Phi_\lambda^t(x_\tau)$ is the solution at time $t + \tau$ if it was x_τ at time τ.

Relation between flow and vector field

If all solutions of a flow are of the form $x(t+\tau) = \Phi^t_\lambda(x_\tau)$ then

$$\frac{d}{dt}x(t+\tau) = \frac{d}{dt}\Phi^t_\lambda(x_\tau)$$

and in particular

$$\frac{d}{dt}x(t+\tau)|_{t=0} = \frac{d}{dt}\Phi^t_\lambda|_{t=0}(x_\tau) := f(x_\tau, \lambda),$$

or more simply

$$\frac{d}{dt}x = \frac{d}{dt}\Phi^t_\lambda|_{t=0}(x) := f(x, \lambda).$$

Example

Consider again the ODE $\frac{dx}{dt} = x$. The flow here is the map $\Phi^t : \mathbb{R} \to \mathbb{R}$ so that $\Phi^t(x)$ is the point where the point $x \in \mathbb{R}$ gets transported to along a solution of the ODE after time t:

$$\Phi^t(x) = \exp(t)x.$$

We observe some elementary geometric features:

- Φ^t has a fixed point 0, since $\Phi^t(0) = 0$ for all t.
- The forward time flow, Φ^t with $t > 0$, expands every interval: $|\Phi^t([a, b])| = |[\exp(t)a, \exp(t)b]| = \exp(t)|[a, b]| > |[a, b]|$

Qualitative changes of the flow: bifurcations

Parameters are important in many applications. They represent quantitative *constant* factors in models. If parameters change value, the quantitative features of the flow (and solutions) generally change. At the same time, it turns out that often important qualitative features do not change much when parameters are varied, unless the parameter values are very special (in the context of the model). We refer to the latter special parameter values as *bifurcation points*.

Example

We consider a variation of our running example by including a parameter:

$$\frac{dx}{dt} = \lambda x, \quad \Phi^t_\lambda(x) = \exp(\lambda t)x$$

We now consider the qualitative features of the flow:

- 0 is a fixed point for the flow, $\Phi^t(0) = 0$, independently of λ.
- if $\lambda = 0$, all points x are fixed by the flow.
- $|\Phi^t([a,b])| = |[\exp(\lambda t)a, \exp(\lambda t)b]| = \exp(\lambda t)|[a,b]|$.

The latter observation implies that we have two qualitatively different parameter regimes:

- if $\lambda < 0$ the forward time flow moves all points towards the fixed point 0
- if $\lambda > 0$ all points are moved away from this fixed point.

We recognize the parameter value $\lambda = 0$ as a *bifurcation point*.