Appendix A

Proof of Poincaré-Bendixson theorem

We here finalize the proof of the (in my view, more complete) Poincaré-Bendixson theorem, building on the (less complete) Poincaré-Bendixson theorem that is presented in §10.5 of HSD:

Theorem A.0.6 (Poincaré-Bendixson)**.** *Suppose that Ω is a nonempty, closed and bounded limit set of a planar flow, then one of the following holds*

- Ω *is an equilibrium point*
- Ω *is a periodic solution*
- Ω *consists of a set of equilibria and connecting orbits between these equilibria.*

Proof. We consider $\Omega = \omega(\mathbf{x})$ for some $\mathbf{x} \in \mathbb{R}^2$. The argument in the case of an α-limit set is similar.

Let $\mathbf{y} \in \omega(\mathbf{x})$ and $\mathbf{z} \in \omega(\mathbf{y})$. In the proof of the Theorem in §10.5 of HSD, it was shown that if $\mathbf{z}$ is not an equilibrium point, then $\omega(\mathbf{y})$ must be a periodic solution and that if $\omega(\mathbf{y})$ is a periodic solution then $\omega(\mathbf{x}) = \omega(\mathbf{y})$ and thus $\omega(\mathbf{x})$ is also a periodic solution.

Now assume that $\mathbf{z} \in \omega(\mathbf{y})$ is an equilibrium point. Then $\omega(y)$ must consist entirely of equilibria since if there is a point $\mathbf{z} \in \omega(\mathbf{y})$ that is not an equilibrium, then we know that $\omega(\mathbf{y})$ is a periodic solution (and in particular contains no equilibrium). Note that since $\mathbf{y} \in \omega(\mathbf{x})$ it follows that $\{\Phi^t(\mathbf{y})\}_{t\in\mathbb{R}} \subset \omega(\mathbf{x})$, where Φ^t denotes the time-t flow. Hence $\alpha(\mathbf{y}) \in \omega(\mathbf{x})$ and for the same reasons as before $\alpha(\mathbf{y})$ must be an equilibrium, since otherwise $\omega(\mathbf{y})$ (and $\omega(\mathbf{x})$) must be a periodic solution. Hence we find that either $\mathbf{y}$ is an equilibrium point, or that $\mathbf{y}$ lies in the intersection between the stable and unstable manifolds of the equilibria $\omega(\mathbf{y})$ and $\alpha(\mathbf{y})$ (i.e. on a connecting orbit between equilibria). □