Chapter 4

Existence and uniqueness of solutions for nonlinear ODEs

In this chapter we consider the existence and uniqueness of solutions for the initial value problem for general nonlinear ODEs. Recall that it is this property that underlies the existence of a flow. We only consider the problem for autonomous ODEs, but note that through (1.1.3) the non-autonomous case follows as a corollary from the autonomous one. We consider the autonomous ODE

$$\frac{d\mathbf{x}}{dt} = f(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^m. \tag{4.0.1}$$

We are interested in solving the initial value problem to find $\mathbf{x}(t)$ satisfying (4.0.1) $\mathbf{x}(t_0) = \mathbf{x}_0$. We note that since (4.0.1) is autonomous a solution to this problem implies the existence of a solution for the initial value problem $\mathbf{x}(\tau) = \mathbf{x}_0$ where $\tau \neq t_0$.

In the case that the vector field f is linear, we have seen in Chapter 2 that the initial value problem has a unique solution. The aim of this chapter is to prove the existence of unique solutions also in the case that f is nonlinear. We will set up the problem in such a way that we obtain the solution by an application of the Contraction Mapping Theorem that was discussed in Chapter 3.

4.1 Picard iteration

The first step we undertake is to reformulate (4.0.1) as an integral equation. By formally integrating (4.0.1) we obtain

$$\mathbf{x}(t) = \mathbf{x}_0 + \int_{t_0}^{t} f(\mathbf{x}(s))ds, \tag{4.1.1}$$

where the integration constant is chosen such that $\mathbf{x}(t_0) = \mathbf{x}_0$. This does not yield an explicit solution, since both the left- and right-hand-side contains a reference to the solution $\mathbf{x}(t)$. It follows that with initial value $\mathbf{x}(t_0) = \mathbf{x}_0$, $\mathbf{x}(t)$ is a solution of (4.0.1) if and only if $\mathbf{x}(t)$ is a solution of (4.1.1): by differentiating (4.1.1) it is immediate that (4.1.1) implies (4.0.1), the implication in the opposite direction follows by the fact that we used the choice of constant (the only freedom available in integrating (4.0.1)) to satisfy the initial value condition $\mathbf{x}(t_0) = \mathbf{x}_0$.

We consider (4.1.1) as the basis of the definition of an operator T on functions u from a closed time interval $[t_0 - a, t_0 + a]$ to $\mathbb{R}^m$ that satisfy $\mathbf{u}(0) = \mathbf{x}_0$:

$$T(\mathbf{u}(t)) = \mathbf{x}_0 + \int_{t_0}^{t} f(\mathbf{u}(s))ds. \tag{4.1.2}$$

We observe that solutions of the initial value problem (4.0.1) with $\mathbf{x}(t_0) = \mathbf{x}_0$ correspond to fixed points of the operator T: if $T(\mathbf{x}(t)) = \mathbf{x}(t)$ then (4.1.1) is satisfied, which in turn implies that $\mathbf{x}(t)$ is a solution of (4.0.1) with initial value $\mathbf{x}(t_0) = \mathbf{x}_0$.

The strategy is to show that when we consider T as defined on a suitable complete metric space, T is a contraction. Then we find a unique fixed point for T, that corresponds to the unique solution of the initial value problem of the ODE.

Before exploring this idea in more detail, let us verify first that this approach may make sense by trying to solve some simple initial value problems for ODEs using iteration of T. This process is known as *Picard iteration*.

Example 4.1.1. Consider

$$\frac{dx}{dt} = rx \quad \text{with } x \in \mathbb{R} \text{ and initial value } x(0) = x_0.$$

We denote the iteration as

$$u_{j+1}(t) = T(u_j(t)) = \mathbf{x}_0 + r\int_{t_0}^{t} u_j(s)ds. \tag{4.1.3}$$

We are interested in $\lim_{n\to\infty} u_n(t)$. As initial condition for the iteration process we choose the constant function $u_0(t) = x_0$ (which is the simples example of a function u_0 that satisfies $u_0(0) = x_0$). Then

$$\begin{aligned}
u_1(t) &= T(u_0(t)) = x_0 + r\int_0^t x_0 ds = x_0(1 + rt),\\
u_2(t) &= T(u_1(t)) = x_0 + r\int_0^t x_0(1 + rs)ds = x_0(1 + rt + \frac{1}{2}(rt)^2),\\
u_3(t) &= T(u_2(t)) = x_0(1 + rt + \frac{1}{2}(rt)^2 + \frac{1}{3!}(rt)^3),\\
u_n(t) &= x_0\sum_{j=0}^{n} \frac{(rt)^j}{j!},
\end{aligned}$$

so that $\lim_{n\to\infty} u_n(t) = x_0 e^{rt}$ which is indeed the unique solution to the initial value problem for this ODE (as we know from Chapter 2).

Let us try the iteration also with another initial function u_0, for instance $u_0 = x_0 + t$. Then

$$
\begin{aligned}
u_1(t) &= T(u_0(t)) = x_0 + r\int_0^t (x_0+s)ds = x_0(1+rt) + \frac{1}{2}t^2,\\
u_2(t) &= T(u_1(t)) = x_0 + r\int_0^t (x_0(1+rs)+\frac{1}{2}s^2)ds = x_0(1+rt+\frac{1}{2}(rt)^2) + \frac{1}{3!}t^3,\\
u_3(t) &= T(u_2(t)) = x_0(1+rt+\frac{1}{2}(rt)^2+\frac{1}{3!}(rt)^3) + \frac{1}{4!}t^4,\\
u_n(t) &= x_0\left(\sum_{j=0}^{n}\frac{(rt)^j}{j!}\right) + \frac{1}{(n+1)!}t^{n+1},
\end{aligned}
$$

so that we find that $u_n(t)$ is equal to the nth order Taylor expansion of the solution plus an additional term $\frac{1}{(n+1)!}t^{n+1}$. Fortunately, when considering any fixed value of t, this term tends to zero as n tends to infinity, so that indeed we obtain as desired $\lim_{n\to\infty} u_n(t) = x_0 e^{rt}$.

Example 4.1.2. Consider the linear ODE

$$
\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \quad \text{with initial condition } \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
$$

We perform Picard iteration with $\mathbf{u}_0(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$:

$$
\begin{aligned}
\mathbf{u}_1(t) &= T(\mathbf{u}_0(t)) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_0^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} ds = \begin{pmatrix} 1 \\ -t \end{pmatrix},\\
\mathbf{u}_2(t) &= T(\mathbf{u}_1(t)) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_0^t \begin{pmatrix} -s \\ -1 \end{pmatrix} ds = \begin{pmatrix} 1-\frac{t^2}{2} \\ -t \end{pmatrix},\\
\mathbf{u}_3(t) &= T(\mathbf{u}_2(t)) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_0^t \begin{pmatrix} -s \\ -1+\frac{s^2}{2} \end{pmatrix} ds = \begin{pmatrix} 1-\frac{t^2}{2} \\ -t+\frac{t^3}{3!} \end{pmatrix},\\
\mathbf{u}_n(t) &= \begin{pmatrix} n\text{th order taylor expansion of } \cos(t) \text{ at } t=0 \\ n\text{th order taylor expansion of } -\sin(t) \text{ at } t=0 \end{pmatrix}.
\end{aligned}
$$

And indeed the solution is

$$
\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \cos(t) \\ -\sin(t) \end{pmatrix}.
$$

We recall that the idea is to find a complete metric space (which is a function space) on which T is a contraction, and thus yielding a unique fixed point that corresponds to the unique solution of the initial value problem for the ODE.

In order to appreciate the choice of function space that we will use in a few moments, let us first explore some examples of initial value problems for ODEs where existence and uniqueness of solutions does not hold.

Example 4.1.3. Let

$$\frac{dx}{dt} = \begin{cases} 1 & \text{if } x < 0 \\ -1 & \text{if } x \geq 0 \end{cases}$$

with $x \in \mathbb{R}$. Consider the initial value problem of this ODE with discontinuous vector field, and initial value $x(0) = 0$. Since $\frac{dx}{dt} = -1$, we observe that x is decreasing but the solution cannot be decreasing since as soon as $x < 0$ we have $\frac{dx}{dt} > 0$ so that it must be increasing. Hence there does not exists a solution with initial value $x(0) = 0$ for this ODE. We note that the property of the vector field that appears to create this problem is the fact that the vector field is discontinous at $x = 0$. So we will not try to prove existence and uniqueness for discontinuous vector fields.

Example 4.1.4. Consider

$$\frac{dx}{dt} = 3x^{2/3}, \quad \text{with } x \in \mathbb{R}$$

and initial value $x(0) = 0$. One (immediately obvious) solution is $x(t) = 0$ for all t. But one readily verifies that there also exists another solution: $x(t) = t^3$, since $\frac{dx(t)}{dt} = 3t^2 = 2(t^3)^{2/3}$. So we here have existence, but not uniqueness of solutions. We observe that although the vector field if continuous, it is not differentiable (as the derivative blows up at $x = 0$) and even not Lipschitz. We will later on insist on the fact that the vector field is Lipschitz (which is a slightly weaker property than continuous differentiability).

Example 4.1.5. Our final example here will illustrate that even if we have existence and uniqueness of solutions for a given initial value problem of an ODE, such solutions may well not exist for all time. For instance, consider the ODE

$$\frac{dx}{dt} = 1 + x^2, \quad \text{with } x \in \mathbb{R}.$$

Then we can integrate this ODE by means of separation of variables:

$$\int \frac{dx}{1+x^2} = \int dt \quad \Leftrightarrow \quad \tan^{-1}(x) = t + c \quad \Leftrightarrow \quad x(t) = \tan(c+t).$$

Hence, despite the fact that for any initial value problem we can find a unique solution, we cannot avoid this solution to blow up to $\pm\infty$ in finite time (when $t + c = \pi/2 \bmod \pi$).

4.2 The Picard-Lindelöf Theorem

We will now prove a theorem about existence and uniqueness of solutions of intial value problems for ODEs that is known as the "Picard-Lindelöf Theorem". We do not present the most general (or strongest) version of this theorem, but a version that admits a straightforward proof using the Contraction Mapping Theorem.

Motivated by the examples of the last Section, we consider solutions that are continuous functions from a finite time-interval $J$to a bounded subset $U \subset \mathbb{R}^m$ (as we want to avoid solutions to blow up). With initial value $\mathbf{x}(t_0) = \mathbf{x}_0$, we will set $J = [t_0 - a, t_0 + a]$ and $U = \overline{B(\mathbf{x}_0, b)}$ (the closed ball in $\mathbb{R}^m$ around $\mathbf{x}_0$ with radius b). We let $C^0(J, U)$ denote the set of continuous functions from J to U. It turns out that

Proposition 4.2.1. *$C^0(J, U)$ is a complete metric space with respect to the metric induced by the* supremum norm

$$d(u, v) = ||u - v||_0 := \sup_{t \in J} |u(t) - v(t)|, \tag{4.2.1}$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^m$: $|\mathbf{x}| = \sqrt{\sum_{i=1}^m x_i^2}$ where $\mathbf{x} = (x_1, \dots, x_m)$.

Proof. It is readily verified that for any Cauchy sequence $u_j \in C^0(J, U)$ it follows that $u_j(t) \in \mathbb{R}^m$ is a Cauchy sequence in $\mathbb{R}^m$ for all $t \in J$. Hence, since $\mathbb{R}^m$ is complete it follows that $C^0(J, U)$ is complete as well. $\square$

The main theorem about existence and uniqueness of solutions follows from the fact that under some mild condition on the time-interval J, the map T defined in (4.1.2) which is at the basis of the Picard iteration is a contraction on this metric space.

Theorem 4.2.2 (Picard-Lindelöf)**.** *Consider the ODE*

$$\frac{d\mathbf{x}}{dt} = f(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^m,$$

with initial value problem $\mathbf{x}(t_0) = \mathbf{x}_0$. Let $U = \overline{B(\mathbf{x}_0, b)}$ and $J = [t_0 - a, t_0 + a]$, where $f : U \to \mathbb{R}^m$ is Lipschitz with Lipschitz constant K, and $|f(\mathbf{x})| \leq M$ for all $\mathbf{x} \in U$, then the initial value problem has a unique solution $\mathbf{x} \in C^0(J, U)$ as long as the time-interval is chosen with a satisfying $0 < a < \min(1/K, b/M)$.

Proof. The aim is to show that with $0 < a < \min(1/K, b/M)$ the map T defined in (4.1.2) is a contraction on the metric space $C^0(J, U)$ with metric (4.2.1). Existence and uniqueness then follows directly by application of the Contraction Mapping Theorem.

First we note that f is continuous because f is Lipschitz. Then f takes a maximum and minimum on U since U is compact (closed and bounded), and hence a finite bound M exists.

In order to make sure that T maps $C^0(J, U)$ into itself we must make sure that a is small enough to guarantee that $T(u(t)) \in U$ for all $t \in J$. We use the bound M to obtain with $t \in J = [t_0 - a, t_0 + a]$ that

$$|T(\mathbf{u}(t)) - \mathbf{x}_0| = \left| \int_{t_0}^t f(\mathbf{u}(s)) ds \right| \leq \left| \int_{t_0}^t |f(\mathbf{u}(s))| ds \right| \leq Ma$$

from which it follows that if $a < b/M$ we have $T(\mathbf{u}(t)) \in B(\mathbf{x}_0, b)$ for all $t \in J$.

It remains to be shown that T is a contraction on $C^0(J, U)$. Consider two elements $u, v \in C^0(J, U)$. Then

$$\begin{aligned} |T(u(t)) - T(v(t))| &= \left| \int_{t_0}^t f(u(s)) - f(v(s)) ds \right| \leq \left| \int_{t_0}^t |f(u(s)) - f(v(s))| ds \right| \\ &\leq K \left| \int_{t_0}^t |u(s) - v(s)| ds \right| \leq aKd(u, v). \end{aligned}$$

Hence, if $aK < 1$ the map T is a contraction on $C^0(J, U)$. □

Theorem 4.2.2 establishes the existence of a flow Φ^t, albeit possibly only on a small time interval. One can actually substantially improve on this result. For instance, one can show that the flow exist for all time if f is Lipschitz on its entire domain (such as is the case when the domain is compact and f is continuously differentiable), but we will not go into the details of such results here.

It is not yet enough to know that a flow map exists, but we would also like to establish some useful properties of the flow such as continuity and differentiability with respect to time and initial conditions. In the special case of flows of linear autonomous ODEs we have already seen in Chapter 2 from the explicit solution that it is continuous and differentiable (actually C^∞) with respect to time and initial conditions. In general, it turns out that also in the nonlinear case the flow is well behaved: if $f : \mathbb{R}^m \to \mathbb{R}^m$ is C^k (k times continuously differentiable) then so is Φ^t. We will only prove a weaker statement, that provides some insight.

Theorem 4.2.3. *Let $O \in \mathbb{R}^m$ be open and suppose $f : O \to \mathbb{R}^m$ is Lipschitz, with Lipschitz constant K. Let $\mathbf{y}(t)$ and $\mathbf{z}(t)$ be solutions of the ODE*

$$\frac{d\mathbf{x}}{dt} = f(\mathbf{x})$$

which remain in O and are defined for $t \in [t_0, t_1]$. Then for all $t \in [t_0, t_1]$ we have

$$|y(t) - z(t)| \leq |y(t_0) - z(t_0)| \exp(K(t - t_0)).$$

It follows from Theorem 4.2.3 that Φ^t is continuous (with respect to its domain and time-variable t). Namely, it follows that $\mathbf{y}(t) \to \mathbf{z}(t)$ for all $t \in [t_0, t_1]$ if $\mathbf{y}(t_0) \to \mathbf{z}(t_0)$. The proof of this theorem involves a famous inequality.

Proposition 4.2.4 (Gronwall's inequality)**.** *Let $u : [0, \alpha] \to \mathbb{R}$ be continuous and non-negative. Suppose $C \geq 0$ and $K \geq 0$ are such that*

$$u(t) \leq C + \int_0^t Ku(s)ds, \quad \forall\, t \in [0, \alpha],$$

then

$$u(t) \leq Ce^{Kt}, \quad \forall\, t \in [0, \alpha].$$

Proof. Suppose first that $C > 0$ and define

$$U(t) = C + \int_0^t Ku(s)ds.$$

Then we have $U(t) \geq u(t)$ and also

$$\frac{dU}{dt}(t) = Ku(t) \;\Rightarrow\; \frac{\frac{dU}{dt}(t)}{U(t)} \leq K,$$

so that, using $U(0) = C$, we obtain

$$\frac{d}{dt}\ln U(t) \leq K \;\Rightarrow\; \ln U(t) \leq \ln U(0) + Kt \;\Rightarrow\; u(t) \leq U(t) \leq Ce^{Kt}.$$

The result with $C = 0$ follows by taking the limit $C \downarrow 0$. $\square$

Proof of Theorem 4.2.3. Define $\mathbf{v}(t) = |\mathbf{y}(t) - \mathbf{z}(t)|$. We use the integral formulation of the initial value problem (4.1.1) to obtain

$$\mathbf{y}(t) - \mathbf{z}(t) = \mathbf{y}(t_0) - \mathbf{z}(t_0) + \int_{t_0}^{t} (f(\mathbf{y}(s)) - f(\mathbf{z}(s)))ds$$

which implies that

$$\mathbf{v}(t) \leq \mathbf{v}(t_0) + \int_{t_0}^{t} |f(\mathbf{y}(s)) - f(\mathbf{z}(s))|ds \leq \mathbf{v}(t_0) + \int_{t_0}^{t} K\mathbf{v}(s)ds.$$

Finally we apply Gronwall's inequality to $\mathbf{u}(t) := \mathbf{v}(t + t_0)$ to obtain

$$\mathbf{u}(t) = \mathbf{v}(t + t_0) \leq \mathbf{v}(t_0) + \int_{t_0}^{t+t_0} |f(\mathbf{y}(s)) - f(\mathbf{z}(s))|ds \leq \mathbf{v}(t_0) + \int_{0}^{t} K\mathbf{u}(s)ds$$

which implies

$$\mathbf{v}(t + t_0) = \mathbf{u}(t) \leq \mathbf{v}(t_0)e^{Kt} \;\Rightarrow\; \mathbf{v}(t) \leq \mathbf{v}(t_0)e^{K(t-t_0)}$$

for all $t \in [t_0, t_1]$. $\square$

Along the same lines one can prove differentiability of the flow to the same degree as existing differentiability of the vector field f. We do not further elaborate on proofs of such facts, which can be found elsewhere.

4.3 Epilogue

After having established the existence and uniqueness for initial value problems in Chapter 2 for linear ODEs by finding a unique explicit solution, in this chapter we have shown that the existence and uniqueness for initial value problems does not always hold, but that it still holds for nonlinear ODEs with some mild regularity properties of the vector field, such as being locally Lipschitz. We have obtained this result without finding explicit solutions (which turns out to be a rather useless aim in this context).

We note that although existence and uniqueness of initial value problems is a quite natural and desirable property from a modeling perspective, it does not usually hold for other types of differential equations (such as partial differential equations).

The existence and uniqueness of initial value problems enables us to study the *dynamics* of an ODE in terms of the *flow* on the phase space generated by its solutions. In the remaining chapters we will focus on the analysis of flows of ODEs from a geometric point of view.