Chapter 1

Introduction

1.1 Ordinary Differential Equations

This course deals with a very important class of differential equations, so-called *ordinary differential equations (ODEs)*. These have the form

$$\frac{d\mathbf{x}}{dt} = f(\mathbf{x}, \lambda, t). \tag{1.1.1}$$

In this equation, $\mathbf{x}$ represents a (set of) variable(s) that describes the state of our system in our *phase space*, and $\lambda \in \mathbb{R}^p$ denote additional (external) parameters. In the context of this course we will always take the phase space to be $\mathbb{R}^m$ for some $m \geq 1$. In this case f is a map from $\mathbb{R}^m \times \mathbb{R}^p \times \mathbb{R}$ to $\mathbb{R}^m$. Such a map is called a *vector field*, as it associates with each point in the phase space $\mathbb{R}^m$ (at constant value of the parameters and time) a vector that represents $\frac{d\mathbf{x}}{dt}$. Unless stated otherwise, we will here always assume that f is *smooth* in its arguments, meaning that derivates of f with respect to all its arguments and to all orders exist and are continuous.

One may think of the vector as representing the *velocity* at the point $\mathbf{x}$ in the phase space. Solutions of (1.1.1) take the form $\mathbf{x}(t, \lambda)$ where λ represents the (fixed) parameters in the vector field f and t the independent variable, which in many models is associated with *time*. If we are not interested in changing the parameters, we often suppress the parameters and denote solutions $\mathbf{x}(t)$ instead of $\mathbf{x}(t, \lambda)$.

The objective is on the one hand to understand the set of solutions $\mathbf{x}(t, \lambda)$ of (1.1.1) for a fixed choice of parameters λ and on the other hand to understand how this set of solutions changes as λ is being varied. Unfortunately, except in special cases, full answers to these questions are well beyond (our and everyone's) reach. So we will address mainly more modest objectives.

The main aim of this course is to develop a geometric point of view to the solutions $x(t)$ of (1.1.1), and then use this geometric point of view to analyze properties of the solutions of ODEs.

ODEs form the backbone of models for so-called *dynamical systems* that describe the evolution of variables characterizing a system in the course of time.

Examples that you may have seen include the description of motion of simple mechanical systems. In the latter case, the equations of motion are often written in the form

$$\frac{d^2\mathbf{x}}{dt^2} = g(\frac{d\mathbf{x}}{dt}, \mathbf{x}, \lambda, t).$$

We call this a *second order* ODE, referring to the highest order derivative of $\mathbf{x}$ that appears.

By introducing the variable $\mathbf{y} = \frac{d\mathbf{x}}{dt}$, we can rewrite this differential equation in the form

$$\left\{ \begin{array}{rcl} \frac{d\mathbf{y}}{dt} & = & g(\mathbf{y}, \mathbf{x}, \lambda, t), \\ \frac{d\mathbf{x}}{dt} & = & \mathbf{y}. \end{array} \right. \tag{1.1.2}$$

Note that the latter set of coupled equations has the general form of (1.1.1): let $\mathbf{z} = (\mathbf{y}, \mathbf{x})$, then we may represent (1.1.2) as $\frac{d\mathbf{z}}{dt} = f(\mathbf{z}, \lambda, t)$ with f defined by the right hand side of (1.1.2).

Example 1.1.1. Newton's equations of motion for a damped harmonic oscillator with mass m, natural frequency ω and damping coefficient μ are

$$m\ddot{x} = -\omega^2 x - \mu\dot{x}.$$

This can be rewritten as

$$\left\{ \begin{array}{rcl} \frac{d\mathbf{y}}{dt} & = & \frac{1}{m}(-\omega^2\mathbf{x} - \mu\mathbf{y}), \\ \frac{d\mathbf{x}}{dt} & = & \mathbf{y}, \end{array} \right.$$

so that the ODE takes the form

$$\frac{d}{dt}\begin{pmatrix} \mathbf{y} \\ \mathbf{x} \end{pmatrix} = f(\begin{pmatrix} \mathbf{y} \\ \mathbf{x} \end{pmatrix}, m, \omega, \mu) = \begin{pmatrix} \frac{1}{m}(-\omega^2\mathbf{x} - \mu\mathbf{y}) \\ \mathbf{y} \end{pmatrix}.$$

where $\lambda = (m, \omega, \mu)$ are the parameters.

In general we say that a differential equation of the form

$$\frac{d^k\mathbf{x}}{dt^k} = g(\frac{d^{k-1}\mathbf{x}}{dt^{k-1}}, \ldots, \frac{d\mathbf{x}}{dt}, \mathbf{x}, \lambda, t),$$

is a kth order ODE. Along the lines of the above discusssion, it follows that such an ODE can always be written in the form of a first order ODE, by introduction of the auxiliary variables $\mathbf{y}_n = \frac{d^{k-n}\mathbf{x}}{dt^{k-n}}$ where $n = 1, \ldots, k-1$:

$$\left\{ \begin{array}{rcl} \frac{d\mathbf{y}_1}{dt} & = & g(y_1, \ldots, \mathbf{y}_{k-1}, \mathbf{x}, \lambda, t), \\ \frac{d\mathbf{y}_2}{dt} & = & \mathbf{y}_1, \\ \vdots & = & \vdots \\ \frac{d\mathbf{y}_{k-1}}{dt} & = & \mathbf{y}_{k-2}, \\ \frac{d\mathbf{x}}{dt} & = & \mathbf{y}_{k-1}, \end{array} \right.$$

In this course we will deal with first order ODEs, motivated by the fact that higher order ODEs can always be rewritten as first order ODEs by the introduction of auxiliary variables.

One should however keep in mind that first order ODEs that can be written as higher order ODEs are a very special kind of first order ODEs. Most first order ODEs cannot be reformulated as a higher order ODE with less dependent variables.

In (1.1.1) we indicated that the vector field f may depend explicitly on the independent variable t. Although we in principle deal with such ODEs, most of the time we will focus on *autonomous* ODEs that do not explicitly depend on this variable.

Definition 1.1.2 (Autonomous and non-autonomous ODEs)**.** In case the vector field f in the ODE (1.1.1) does not explicitly depend on the independent variable t, then we call the ODE *autonomous*. Alternatively, if the vector field depends explicitly on the independent variable t, then we call the ODE *non-autonomous*.

It is sometimes useful to observe that one may view any autonomous ODE as an autonomous ODE on an extended phase space. This phase space is constructed from the original one with a one-dimensional extension that takes account of the time-variable:

$$\frac{d\mathbf{x}}{dt} = f(\mathbf{x}, \lambda, t) \quad \Leftrightarrow \quad \begin{cases} \frac{d\mathbf{x}}{dt} &= f(\mathbf{x}, \lambda, \tau) \\ \frac{d\tau}{dt} &= 1 \end{cases} \tag{1.1.3}$$

1.2 Geometric picture: flow

A very important property of ODEs is the *existence and uniqueness of solutions*. We will describe this property in much more detail later, but it roughly refers to the fact that if we specify the state of a system described a point $\mathbf{x}_0$ in the phase space (at a given moment in time t, say $t = 0$), then there is a unique solution $\mathbf{x}(t)$ of the ODE so $\mathbf{x}(0) = \mathbf{x}_0$. In other words, each *state* of the system can be attained, and has a unique past and a unique future (for some time-interval including $t = 0$; we will see later that there are some instances in which solutions may blow up in finite time, in which case they are not defined for all time, obviously).

Example 1.2.1. Consider the ODE $\frac{dx}{dt} = x$, $x \in \mathbb{R}$. With initial condition $x(\tau) = y$ this has the unique solution $x(t) = y\exp(t - \tau)$. Existence and uniqueness means that through every point y of the phase space $\mathbb{R}$ there exists a unique solution $x(t) = \exp(t - \tau)y$ that intersects y at $t = \tau$.

At first sight existence and uniqueness may seem an obviously desirable property of any model, but there are many differential equations (other than ODEs) that do not have this property. For some differential equations, certain *initial conditions* lead to contradictions, meaning that not every state of the system is attainable. And attainable states may not have unique "past and futures" in terms of the independent variables.

In order to prove the fact that ODEs possess the above mentioned properties, we will need to develop some analytical tools (later in the course).

An important consequence of the existence and uniqueness of solutions is that we obtain the possibility to associate with the solutions of ODEs a dynamical process that we call a *flow*. The basic idea is the following: instead of thinking of solutions of ODEs as functions $x(t)$ with $x : \mathbb{R} \to \mathbb{R}^m$ satisfying (1.1.1) with some initial (boundary) condition, we take the point of view that the solutions of the ODE describe a *transformation of the phase space*. This transformation of the phase space represents the *dynamics* of the system as modeled by the ODE, describing the future (and past) of states of the system. In the autonomous case this transformation does depend on the length of the time-interval, and in the non-autonomous case there is an additional dependence on the initial time. We denote the flow map (from time t_0 to t_1 at parameter value λ) $\Phi_\lambda^{t_1,t_0} : \mathbb{R}^m \to \mathbb{R}^m$: $\Phi_\lambda^{t_1,t_0}(\mathbf{x})$ is the solution at time t_1 if it was $\mathbf{x}$ at time t_0. In case the ODE is autonomous the flow map only depends on the time interval $t = t_1 - t_0$ and we denote the flow map as Φ_λ^t. Often we also suppress the subscript λ if we are not concerned with explicit issues considering the dependence of the flow on parameters.

From the definition of the flow map above, it follows by definition of the derivative that

$$\begin{aligned} \frac{d\mathbf{x}}{dt}(t_0) &= \lim_{t_1\to t_0} \frac{\mathbf{x}(t_1)-\mathbf{x}(t_0)}{t_1-t_0} = \lim_{t_1\to t_0} \frac{\Phi_\lambda^{t_1,t_0}(\mathbf{x}(t_0))-\mathbf{x}(t_0)}{t_1-t_0} \\ &= \lim_{h\to 0} \frac{\Phi_\lambda^{t_0+h,t_0}(\mathbf{x}(t_0))-\Phi_\lambda^{t_0,t_0}(\mathbf{x}(t_0))}{h} \qquad (1.2.1) \\ &= \frac{d}{dt}\Phi_\lambda^{t,t_0}|_{t=t_0}(\mathbf{x}(t_0)) = f(\mathbf{x}(t_0),\lambda,t_0). \qquad (1.2.2) \end{aligned}$$

In the autonomous case the final part of this expression simplifies to

$$\frac{d}{dt}\Phi_\lambda^t|_{t=0}(\mathbf{x}) = f(\mathbf{x},\lambda), \qquad (1.2.3)$$

since then $\Phi_\lambda^t(\mathbf{x}) = \mathbf{x}(t)$ with $\mathbf{x}(0) = \mathbf{x}$. In this (autonomous) case the flow maps $\{\Phi_\lambda^t \mid t \in \mathbb{R}\}$ form a group with the relations

(i) $\Phi_\lambda^t \circ \Phi_\lambda^s = \Phi_\lambda^{t+s}$,

(ii) $\Phi_\lambda^0 = Id$ (identity map),

(iii) $(\Phi_\lambda^t)^{-1} = \Phi_\lambda^{-t}$.

Definition 1.2.2 (Flow)**.** A *flow* is a set of transformations $\{\Phi^t\}_{t\in\mathbb{R}}$, where $\Phi^t : \mathbb{R}^m \to \mathbb{R}^m$ is continuous, and depends continuously on the parameter t, satisfying the group properties (i), (ii) and (ii) above[1].

Remark 1.2.3. This definition of flow is appropriate for the context of ODEs in this course. As the flow may not exist for all t, one also talks of a flow when the transformations are only defined in a finite time-interval $t \in [a, b]$.

Remark 1.2.4. In the non-autonomous case, the set of transformation Φ_λ^{t,t_0} does not form a group, but a closely related object, known as a *groupoid*. We note that nonautonomous systems can always be viewed as autonomous one, using an extended phase space as indicated in (1.1.3).

[1] We suppressed the parameter dependence in the notation.

Example 1.2.5. Consider the ODE of Example 1.2.1 above. The flow Φ^t for this example is (by definition) a map $\Phi^t : \mathbb{R} \to \mathbb{R}$ so that $\Phi^t(y)$ is the point where the point $y \in \mathbb{R}$ gets transported to along a solution of the ODE after time t. The flow map in this case does not explicitly depend on the initial time since the ODE is autonomous. In this case, $\Phi^t : \mathbb{R} \to \mathbb{R}$ is thus defined as $\Phi^t(y) = \exp(t)y$.

From the flow, we observe some elementary geometric features. The map Φ^t has a fixed point 0, since $\Phi^t(0) = 0$ for all t. Also, the forward time flow, Φ^t with $t > 0$, expands every interval: $|\Phi^t([a,b])| = |[\exp(t)a, \exp(t)b]| = \exp(t)|[a,b]| > |[a,b]|$

It turns out that the smoothness of the vector field is closely related to the smoothness of the flow: smooth vector fields give rise to smooth flows. Continuity of the flow leads to the observation that the closer together we choose initial conditions, the closer together the trajectories through these initial conditions stay in forward and backward time. This however, does not prevent solutions starting at almost identical initial conditions having completely different futures after sufficiently long times. The latter phenomenon was recognized for the first time by the French mathematician Henri Poincaré towards the end of the 19th century when he was studying the problem of the stability of the solar system (in classical mechanics). In today's terminology, this phenomenon is known as *sensitive dependence on initial conditions* and is intimately related with the phenomenon of *chaotic dynamics*. Poincaré led the development of switching the emphasis from the search for exact (closed form) expressions of solutions of ODEs to the geometric properties of the flows that they induce. It turns out that in most cases it is impossible to find closed form expressions for solutions of ODEs. In fact, even if one finds such expressions one may still lack essential insight in the geomeric properties of the flow.

Following the initial ideas of Poincaré, the theory of dynamical systems (also known as the geometric theory of ordinary differential equations) has been developed over the last century, in particular since the 1960s when computers became available.

1.3 Variation of parameters and bifurcations

Parameters are important in many applications. They represent quantitative *constant* factors in models. It turns out that often the behaviour of solutions depends on the values of the parameters. In order to understand this, it is important to consider to study how a (small) change of the parameters changes the solutions of a model. In the context of ODEs, where the solutions can be represented in terms of the flow, we thus ask how the flow changes if parameters are changed. Clearly, if parameters change value, the quantitative features of the flow (and solutions) do change. On a more qualitative level it turns out that often important qualitative features do not change much when parameters are varied, unless the parameter values are very special (in the context of the model). We refer to the latter special parameter values as *bifurcation points*.

Example 1.3.1. We consider a variation of Example 1.2.1 by introducing a parameter $\lambda \in \mathbb{R}$: $\frac{dx}{dt} = \lambda x$, $x \in \mathbb{R}$. With initial condition $x(\tau) = y$ this has the unique solution $x(t) = y \exp(\lambda(t - \tau))$.

We now consider the qualitative features of the flow discussed in Example 1.2.5. We observe that 0 is a fixed point for the flow, $\Phi^t(0) = 0$, independently of λ. But we also note that if $\lambda = 0$ actually all points x are fixed by the flow, since in that case the flow map is equal to the identity map. If we examine how the flow acts on intervals, we observe that $|\Phi^t([a, b])| = |[\exp(\lambda t)a, \exp(\lambda t)b]| = \exp(\lambda t)|[a, b]|$. This means that the forward time flow expands intervals if $\lambda > 0$ but that it contracts all intervals (eventually to the fixed point 0) if $\lambda < 0$. This is a clear *qualitative* difference. We thus observe two qualitatively different regimes: if $\lambda < 0$ the forward time flow moves all points towards the fixed point 0 whereas if $\lambda > 0$ all points are moved away from this fixed point. We recognize the parameter value $\lambda = 0$ as a bifurcation point.