M2AA1: How to study for the exam?

The exam will consist of four (multi-part) questions, with roughly the following focus:

Question 1 theory bookwork (literally from course notes)

Question 2 Flow of linear ODE and associated linear algebra: eigenvalues, (generalized) eigenspaces, properties of $\exp(At)$, Jordan Normal Form, Jordan-Chevalley Decomposition, projections. The question will concern the flow of a linear ODE in $\mathbb{R}^3$, so be prepared (and do not only think about planar examples).

Question 3 Flow near an equilibrium of planar vector field: hyperbolicity, stability, Lyapunov function, bifurcation, Poincaré return map, transversality, Implicit Function Theorem.

Question 4 Qualitative analysis of a (biologically motivated) planar ODE: equilibria and their stability, nullclines, basin of attraction, phase portrait, Poincaré-Bendixson Theorem (and its consequences).

Question 1 entirely concerns bookwork (and answers can be found literally in the course notes; questions will cover a variety of unrelated topics). Questions 2,3, and 4 are designed to test your understanding of the material and the answers to these questions generally CANNOT be found literally in the notes.

Apart from learning the definitions and main results with their proofs, one is strongly encouraged investing time in trying to understand the course material as the major part of the exam is designed to test your understanding rather than your memory. The exercises should be of good help.

Below I present a summary of the course which hopefully helps you with studying for the exam. I also discuss which exercises on problem sheets should be studied carefully (and point out a few that you may skip). The exam will only concern autonomous ODEs, so please do not worry about non-autonomous ODEs (that are occasionally mentioned in the notes and exercises).

1 Theory:

You are expected to be able to reproduce the following definitions, and statements and proofs of theorems/propositions concerning the following:

1. Introduction

- What is an ODE, and its order.
- Flow: its definition, meaning and properties

2. Linear ODEs

- Linear vector field, linear ODE (and in general definition of linearity).
- The flow of a linear ODE is linear.
- Existence and uniqueness of solutions to initial value problem for linear ODEs.
- Exponential of matrix. Flow of linear ODE in terms of exponential. Existence and uniqueness for linear ODE.
- Algebraic and geometric multiplicity of eigenvalues of matrix. Eigenspaces and generalized eigenspaces.
- Real Jordan normal form (inclusive of derivation in simple cases).
- Jordan-Chevalley decomposition into semi-simple and nilpotent parts (definition, but not the proof of the uniqueness of this decomposition) and application to the evaluation of flow of linear ODE
- definition of stable, unstable and center subspaces for a linear map (matrix).

- Lyapunov stability, asymptotic stability.
- solution curves of ODEs are everywhere tangent to the vector field.

3. Contractions

- definition of metric, metric space, Cauchy sequence, complete metric space, exponential convergence, contraction
- basic familiarity (in context of course material) with open, closed, bounded sets, neighbourhood, etc
- contraction mapping theorem (statement AND proof), derivative test (proof only in one dimensional case), Newton's method
- inverse an implicit function theorems (do not learn proofs by heart)
- transversality (primarily of curves in $\mathbb{R}^2$ and curves and surfaces in $\mathbb{R}^3$).
- persistence of transversal intersections

4. Existence and uniqueness

- integral formulation of initial value problem of ODE
- method of Picard iteration
- examples of ODEs without uniqueness of solution to initial value problem, examples of ODEs for which solutions to initial value problems do not exist or do not exist for all time.
- Picard-Lindelöf theorem (WITHOUT proof of the fact that the function space on which one defines the Picard iteration is a complete metric space; this may be considered as a given fact)
- Exponential bound on separation of solutions for Lipschitz vector fields
- Gronwall's inequality

5. Flow near equilibria

- equilibria of ODEs appear typically isolated, and are typically hyperbolic
- Hartman-Grobman theorem (no proof)
- stable and unstable manifold theorem (no proof)
- contrasting hyperbolic equilibria versus non-hyperbolic equilibria
- fold bifurcation (in $\mathbb{R}$)
- Hopf bifurcation (in $\mathbb{R}^2$) [no proof]
- Lyapunov functions and related theorems

6. Limit sets (Chapter 10 of Hirsch, Smale and Devaney; sections 10.1, 10.2, 10.3, 10.4, 10.5, 10.6)

- α- and ω-limit sets
- local section and flow box in neighbourhood of non-equilibrium point
- Poincaré return map (including continuity and differentiability as inherited from flow)
- monotonicity property of return maps to local sections for planar flows
- Poincaré-Bendixson Theorem for planar flows and several consequences as in §10.6 of HSD.

7. Applications (Chapters 11 of Hirsch, Smale and Devaney; sections 11.1, 11.2, and 11.3)

- basic familiarity with the modelling of infectious diseases, predator-prey systems , and competitive species

Problem sheets

Please take the following into account when studying the problem sheets:

- Sheet 1: Important: 1,2,3,4,5,6,7,8
  Less important: 9
- Sheet 2: Important: 2,3 (but without estimation of constant c), 5, 6 (but do not learn proof by heart),7
  Less important: 1,4 (do not learn formula or proof by heart)
- Sheet 3: Important: 1,4,5,6,7,8 (but only in $\mathbb{R}^2$ and $\mathbb{R}^3$)
  Less important: 2,3,9
- Sheet 4: Important: 1,2,3,5,7,10,11,12
  Less important: 4
- Sheet 5: Important: 2,3,4,5
  Less important: 1
- Sheet 6: Important: 3,4,5,6,7,9,10
  Less important: 1,2,8
- Sheet 7: Important: all

Of course, the progress tests and last year's exam (mostly already incorporated in the problem sheets) are also useful.

Additional exam preparation: For more exercises, see for instance HSD, but also books by Chicone and Perko (see blog for references).