M2AA1 progress test 16 March 2009, 16:00-17:00

Please attempt all parts of the questions (since they are often unrelated).

1. Consider the ODE

$$\frac{dx}{dt} = f(x, \lambda) = \sin(\lambda) + \lambda\cos(x) + x^2,$$

with $x \in \mathbb{R}$, and $\lambda \in \mathbb{R}$ a parameter.

(a) (i) Use the Implicit Function Theorem to show that the equilibrium $x = 0$ at parameter value $\lambda = 0$ lies on a (locally unique) curve of equilibria in the (λ, x)-plane.

(ii) Show that this curve may be approximated as $\lambda(x) = -\frac{1}{2}x^2 + O(x^3)$. Sketch the corresponding bifurcation diagram (of equilibria) in the (λ, x)-plane.

(iii) Sketch the phase portraits of the flow (near $x = 0$) in the cases $\lambda < 0$, $\lambda = 0$ and $\lambda > 0$.

(b) (i) Let $h(\lambda) := f(0, \lambda)$. Consider the graphs $z = h(\lambda)$ and $z = 0$ in the two-dimensional (λ, z)-plane. Show that these graphs have a transverse intersection in the point $(\lambda, z) = (0, 0)$. Discuss how the "persistence" of transverse intersections relates to the application of the Implicit Function Theorem in part (a)(i).

(ii) Consider the set of equilibria (x, λ) (such that $f(x, \lambda) = 0$), as the intersection of the graphs $z = 0$ and $z = f(x, \lambda)$ in $\mathbb{R}^3$ (with coordinates (x, λ, z)). Show that these graphs intersect transversely in $(x, \lambda, z) = (0, 0, 0)$, and use the dimension formula for transverse intersections to explain why the set of equilibria of f near $(x, \lambda) = (0, 0)$ is one-dimensional (i.e. a curve).

2. Consider the ODE

$$\begin{cases} \dot{x}_1 &=& -2x_2 + x_2x_3 - \varepsilon x_1^3, \\ \dot{x}_2 &=& x_1 - x_1x_3 - \varepsilon x_2^3, \\ \dot{x}_3 &=& x_1x_2 - \varepsilon x_3^3, \end{cases}$$

where $\varepsilon \in \mathbb{R}$ is a non-negative parameter (i.e. $\varepsilon \geq 0$).

(a) Show that $(x_1, x_2, x_3) = (0, 0, 0)$ is a non-hyperbolic equilibrium point.

(b) Determine the linear approximation of this ODE near the equilibrium point and describe its flow.

(c) Show that $V(x_1, x_2, x_3) = x_1^2 + 2x_2^2 + x_3^2$ is a Lyapunov function for this ODE.

(d) Show that $(0, 0, 0)$ is asymptotically stable if and only if $\varepsilon > 0$.

answers

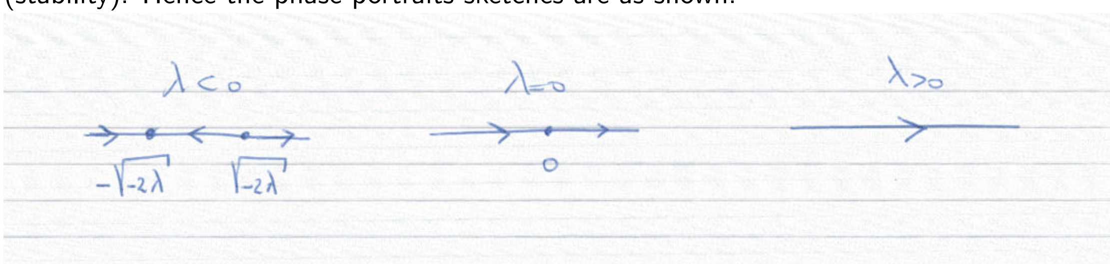
1. (a) (i) [5 pts] $\frac{\partial}{\partial\lambda} f(0,0) = 1$ hence by the IFT there exists unique $\lambda(x)$ with x close to 0 and $\lambda(0) = 0$ such that $f(x, \lambda(x)) = 0$.

(ii) [5 pts] Either by substituting $\lambda(x) = ax + bx^2 + O(x^3)$ and solving $f(x, \lambda) = 0$ up to degree 2:

$$ax + bx^2 + (ax + bx^2) + x^2 + O(x^3) = 0 \;\Leftrightarrow\; a = 0,\; b = -\frac{1}{2}.$$

Or by differentiation: $\frac{d}{dx} f(x, \lambda(x))|_{x=0} = 2\lambda'(0) = 0 \Rightarrow \lambda'(0) = 0$ and $\frac{d^2}{dx^2} f(x, \lambda(x))|_{x=0} = 0$ implying (after some writing out) that $\lambda''(0) = -1$.

(iii) [6 pts] At $\lambda = -\frac{1}{2}x^2 + O(x^3)$ we have $\frac{\partial}{\partial x} f(0,0) = 2x + O(x^3)$ so that the equilibria with small $|x|$ have positive derivative if $x > 0$ (instability) and negative derivative if $x < 0$ (stability). Hence the phase portraits sketches are as shown.



(b) (i) [4 pts] $h(0) = 0$ so the graphs intersect and since $\frac{d}{d\lambda} h(0) = \frac{\partial}{\partial\lambda} f(0,0) = 1 \neq 0$, the tangent to the graph of h (of the form $(1, \frac{d}{d\lambda} h(0))^T$) is linearly independent from the tangent to the graph $z = 0$ (with tangent $(1,0)$). A key result of transversality is that transverse intersections locally persist, i.e. have a unique continuation if f is changed in a smooth way. So if we here consider changing x away from 0, it is like perturbing the function $h(\lambda) = f(0, \lambda)$ to the function $\widetilde{h}(\lambda) = f(x, \lambda)$, so that the solution $\lambda = 0$ will change to a locally unique solution $\lambda(x)$ near 0, which is in line with the conclusion of the IFT.

(ii) [4 pts BONUS] The intersection is transverse if the tangent vectors to the graphs of f and 0 span the $\mathbb{R}^3$. As the plane $z = 0$ has tangent vectors in the vector space generated by $(1,0,0)^T$ and $(0,1,0)^T$, we mainly need to show that there is a tangent vector with nonzero z-component in the tangent space to the graph of f in $(0,0,0)$. The vector $(0, 1, \frac{\partial}{\partial\lambda} f(0,0))$ is such a tangent vector. Hence these graphs intersect transversely. The dimension formula for transverse intersections asserts that $\dim(\text{graph of } f) + \dim(\text{graph of } 0) - \dim(\mathbb{R}^3) = \dim((\text{graph of } f) \cap (\text{graph of } 0))$ so that the latter is equal to $2 + 2 - 3 = 1$.

2. (a) [5 pts] rhs is zero if $x_1 = x_2 = x_3 = 0$ thus $(0,0,0)$ is equilibrium.

$$Df(0,0,0) = \begin{pmatrix} 0 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

So $Df(0,0,0)$ has eigenvalues $0, \pm\sqrt{2}i$ all on the real axis so equilibrium is not hyperbolic.

(b) [5 pts] $\dot{\mathbf{x}} = Df(0,0,0)\mathbf{x}$. All solutions have x_3 constant and the x_1, x_2 variables oscillate periodically (with period $\sqrt{2}\pi$) around $x_1 = x_2 = 0$. So we have a line of equilibria and around that periodic solutions in all planes x_3 =constant.

(c) [4 pts] First $V(\mathbf{x}) \geq 0$ and $V(\mathbf{x}) = 0$ iff $\mathbf{x} = 0$. Moreover,

$$\begin{aligned} \frac{d}{dt}V(\mathbf{x}) &= 2x_1\dot{x}_1 + 4x_2\dot{x}_2 + 2x_3\dot{x}_3 \\ &= 2x_1(-2x_2 + x_2x_3 - \varepsilon x_1^3) + 4x_2(x_1 - x_1x_3 - \varepsilon x_2^3) + 2x_3(x_1x_2 - \varepsilon x_3^3) \\ &= -\varepsilon(2x_1^4 + 4x_2^4 + 2x_3^4) \leq 0 \ \text{ since } \varepsilon \geq 0. \end{aligned}$$

(d) [4 pts] If $\varepsilon > 0$ then $\frac{d}{dt}V(\mathbf{x}) < 0$ for all $\mathbf{x} \neq 0$ and hence (by thm of course) $\mathbf{x} = 0$ is asymptotically stable.
[2 pts] If $\varepsilon = 0$ then $\dot{V}(\mathbf{x}) = 0$ for all $\mathbf{x}$ so that solution never escape from a level set, ie $V(\mathbf{x}(t)) = V(\mathbf{x}(0))$ for all t. This in turn implies that solutions cannot converge to the equilibrium (with $V = 0$) unless the initial condition is on the equilibrium (unique point with $V = 0$). So if $\varepsilon = 0$ the equilibrium is Lyapunov stable (since V is Lyapunov function) but in that case the equilibrium is not asymptotically stable.