M2AA1 progress test 9 February 2009, 16:00-17:00

Please attempt all parts of the question (since they are often unrelated).

. Consider the ODE

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}, \tag{1}$$

with $A \in gl(2,\mathbb{R})$ (2×2 matrix with real coefficients) and $\mathbf{x} \in \mathbb{R}^2$, whose flow $\Phi^t : \mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$\Phi^t(\mathbf{x}) = (\cos(3t)I + \sin(3t)P)\,\mathbf{x},$$

where

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad P = \begin{pmatrix} \frac{4}{3} & -\frac{5}{3} \\ \frac{5}{3} & -\frac{4}{3} \end{pmatrix}.$$

(a) Show that (for all values of $t \in \mathbb{R}$) the flow map $\Phi^t : \mathbb{R}^2 \to \mathbb{R}^2$ is linear.

(b) (i) Determine the eigenvalues λ_1 and λ_2 of A.

(ii) Determine P_1 and P_2 so that the following decomposition holds

$$\Phi^t(\mathbf{x}) = (\exp(\lambda_1 t)P_1 + \exp(\lambda_2 t)P_2)\,\mathbf{x}.$$

where λ_1 and λ_2 denote the eigenvalues of A (as in part (i)).

(iii) Discuss and demonstrate the properties of P_1 and P_2 that follow directly from the decomposition given in part (ii). [Hint: note that $P^2 = -I$.]

(c) (i) Calculate A.

(ii) Determine the real Jordan form and Jordan-Chevalley decomposition of A.

(d) (i) Show that, apart from the equilibrium $(0,0)^T$, all solutions of the ODE are periodic with the same period. Recall that a solution $\mathbf{x}(t)$ is periodic with period T if $T > 0$ is smallest such that $\mathbf{x}(t) = \mathbf{x}(t+T)$.

(ii) Show that one of the periodic solutions has the form

$$\mathbf{x}(t) = \begin{pmatrix} 3\cos(3t) + 4\sin(3t) \\ 5\sin(3t) \end{pmatrix}.$$

(iii) Is the equilibrium Lyapunov stable? Motivate your answer.

Answers: Please divide total [max 40] by 2 (and round up) to get final mark [max 20].

. (a) [4] Linearity of Φ^t means that $\Phi^t(a\mathbf{x} + b\mathbf{y}) = a\Phi^t\mathbf{x} + b\Phi^t\mathbf{y}$ for all $a, a \in \mathbb{R}$ and all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$. The property follows directly from the fact that I and P are linear maps, and Φ^t is a linear combination of I and P. [Alternatively, linearity of Φ^t can also be deduced from the fact that $\Phi^t = \exp(At)$ where A is linear.]

(b) (i) [4] From the way Φ^t depends on t we see that the eigenvalues are $\pm 3i$. [Alternatively, one can also explicitly find A, see (c)(i), and then compute the eigenvalues.]

(ii) [4] Let $\lambda_1 = 3i$ and $\lambda_2 = \overline{\lambda_1}$. Use the fact that $\cos(3t) = \frac{1}{2}e^{3it} + \frac{1}{2}e^{-3it}$ and $\sin(3t) = \frac{1}{2i}e^{3it} - \frac{1}{2i}e^{-3it}$ we obtain

$$P_1 = \frac{1}{2}I - \frac{i}{2}P, \quad P_2 = \frac{1}{2}I + \frac{i}{2}P.$$

(iii) [6] P_1 is the projection to the complex eigenspace associated with eigenvalue λ_1, in the direction of the complex eigenspace associated with eigenvalue λ_2. We thus should have $P_1P2 = P_2P_1 = 0$ (zero matrix) (which is equivalent to $\ker P_1 = \mathrm{range}P_2$ and $\ker P_2 = \mathrm{range}P_1$) and $P_1^2 = P_1$, $P_2^2 = P_2$. Moreover, $P_1 = \overline{P_2}$ (since the flow is real), which is immediate from (ii).

$$P_1P_2 = P_2P_1 = \frac{1}{4}(I - iP)(1 + iP) = \frac{1}{4}(I + P^2) = 0.$$

$$P_1^2 = \frac{1}{4}(I - iP)^2 = \frac{1}{4}(I - P^2 - 2iP) = P_1, \quad P_2^2 = \frac{1}{4}(I + iP)^2 = \frac{1}{4}(I - P^2 + 2iP) = P_2.$$

(c) (i) [4] Since $\frac{d}{dt}\Phi^t|_{t=0} := \frac{d}{dt}\exp(At)|_{t=0} = A$ it follows that

$$A = \frac{d}{dt}\Phi^t|_{t=0} = 3P = \begin{pmatrix} 4 & -5 \\ 5 & -4 \end{pmatrix}.$$

(ii) [4] A has no eigenvalues with multiplicity > 1 so is semi-simple and hence its J.-C.-decomposition is trivial: $A = A + 0$. As it has eigenvalues $\pm 3i$, its real Jordan form is $\begin{pmatrix} 0 & -3 \\ 3 & 0 \end{pmatrix}$ (or $\begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix}$).

(d) (i) [4] One of many correct answers: in Jordan normal form the flow map Φ^t has the form of a rigid rotation around the origin over an angle $3t$. The coordinate transformation does not affect the period of a periodic solution, hence all solutions (except equilibrium at origin) are periodic with period $2\pi/3$.

(ii) [5] From the given expression we find $\mathbf{x}(0) = (3, 0)^T$. One directly computes that indeed with this initial condition we have $\Phi^t(3, 0)^T = \mathbf{x}(t)$ as in the given expression.

(iii) [5] By linearity of the flow, the phase space is foliated by periodic solutions (with size scaling linearly with initial condition) all encircling the origin.
For Lyapunov stability we need to show that for each neighbourhood U of $\mathbf{x}_0$ there exists $V_1 \subset V_0 \subset U$ such that the (positive time) solution through any initial condition in V_1 does not leave V_0. For any U, we can find a periodic solution inside U that encircles the origin. Let $V_0 = V_1$ be the area encircled by this periodic orbit. Then the orbit through any initial condition inside V_0 will not leave V_0. Thus indeed the equilibrium is Lyapunov stable.