M2AA1 Differential Equations
Exercise sheet $\boxed{7}$ answers

1. [HSD, chap 11, q2] In the context of pred-prey we focus on $x, y \geq 0$. We note that the x- and y-axis are flow invariant, but the flow on the y-axis has the problem that $\lim_{x \downarrow 0} \dot{y} \to -\infty$. The nullclines are

$$\begin{aligned} \dot{x} = 0 \quad &: \quad x = 0 \;\text{ or }\; x + y = 1, \\ \dot{y} = 0 \quad &: \quad y = 0 \;\text{ or }\; x = y. \end{aligned}$$

The equilibria are $(0,0)^T$, $(1,0)^T$ and $(1/2,1/2)^T$. The derivative (Jacobian) of the vector field is

$$Df\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 1-2x-y & -x \\ y^2/x^2 & 1-2y/x \end{pmatrix},$$

so that

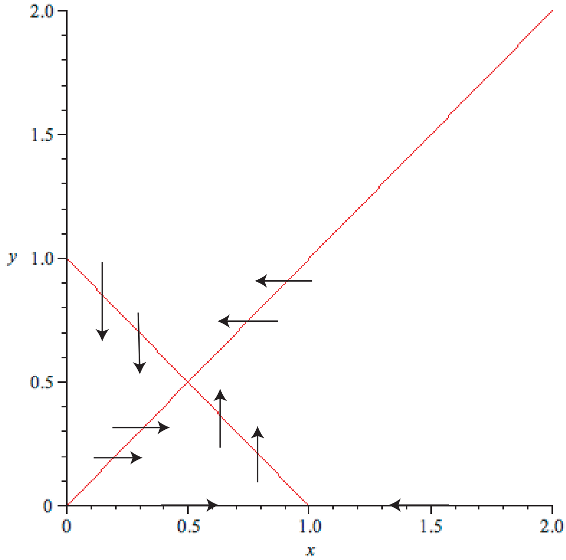
$$Df\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}, \quad Df\left(\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}\right) = \begin{pmatrix} -1/2 & -1/2 \\ 1 & -1 \end{pmatrix}.$$

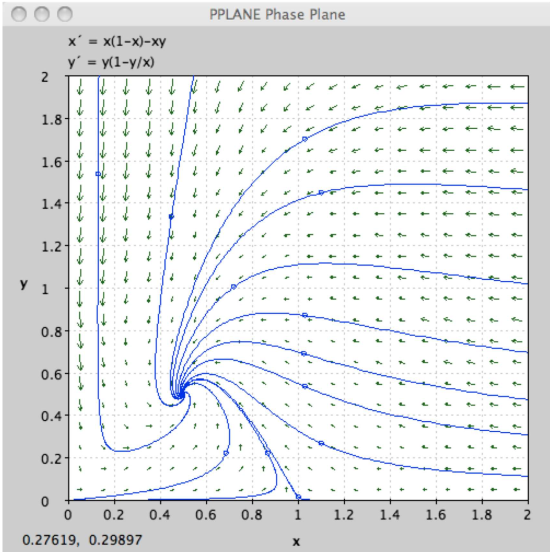
The vector field is not differentiable in $(0,0)^T$ but we note that as the x-axis is an unstable manifold of the equilbrium and the flow is "infinitely" attractive on the y-axis, in quadrant $x, y > 0$ the flow near the equilibrium $(0,0)^T$ looks like as if $(0,0)^T$ is of saddle type. The equilibrium $(1,0)^T$ is a (normal) saddle for which the x-axis serves as stable manifold.

The equilibrium $(1/2,1/2)^T$ is an asymptotically stable focus (calculate the eigenvalues of the jacobian and notice that they are complex with negative real part).

From the flow through the nullclines (see sketch), we see that all initial conditions in the quadrant $x, y > 0$ are attracted to a neighbourhood of $(1/2,1/2)^T$ (for instance the box $[0,3/2] \times [0,3/2]$ is forward flow-invariant and all solutions starting outside this box eventually end up inside this box).

In order to conclude whether or not all solutions converge to the stable equilibrium $(1/2,1/2)^T$ or to a (possibly small) periodic solution that encircles $(1/2,1/2)^T$, we may use the fact that the nullclines in this example are such that we can argue the following. Consider a flow orbit that inside the square $(0,1) \times (0,1)$ intersecting a nullcline, and denote the distance to the point $(1/2,1/2)$ as r. Then the intersection with the next nullcline that this solution intersects will be at a distance $r' < r$. For instance, a solution intersects the line segment between the points $(1/2,1/2)^T$ and $(1,0)^T$ then while travelling to the line segment between the points $(1/2,1/2)^T$ and $(1,1)^T$ the solution curve will move to the left, thereby causing the distance between the intersection point of the solutioon with the line segment between the points $(1/2,1/2)^T$ and $(1,1)^T$ to lie closer to $(1/2,1/2)^T$. This argument holds at each step and for each orbit. Hence, there are no periodic solutions encircling $(1/2,1/2)^T$ and thus all solutions converge to $(1/2,1/2)^T$. [You should note that we can here make this argument because of the special form of the nullclines (with slope 1 and -1). In general we often cannot rule out the existence of periodic solutions just from an analysis using the nullclines.]





2. [HSD, chap 11, q3] The nullclines are

$$\begin{aligned} \dot{x} = 0 \quad &: \quad x = 0 \ \text{ or } \ y = \tfrac{1}{a}(1-x^2), \\ \dot{y} = 0 \quad &: \quad y = 0 \ \text{ or } \ y = 1. \end{aligned}$$

We have equilibria at $(0,0)^T$, $(0,1)^T$, $(0,\pm 1)^T$, and $(\pm\sqrt{1-a}, 1)^T$ (the latter only if $1-a \geq 0$). Denoting the vector field as f, the Jacobian at $(x,y)^T$ is given by

$$Df\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 1-2x-\frac{ay}{(x+1)^2} & -\frac{ax}{1+x} \\ 0 & 1-2y \end{pmatrix},$$

and in the equilibria

$$Df\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad Df\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} -1 & -\frac{x}{2} \\ 0 & 1 \end{pmatrix},$$

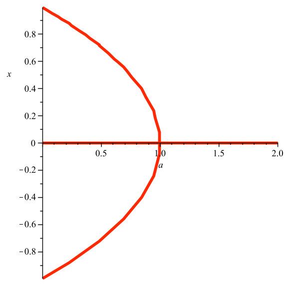
$$Df\left(\begin{pmatrix} \pm\sqrt{1-a} \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 1 \mp 2\sqrt{1-a} - \frac{a}{(1\pm\sqrt{1-a})^2} & \mp\frac{a\sqrt{1-a}}{1\pm\sqrt{1-a}} \\ 0 & -1 \end{pmatrix}.$$

Thus $(0,0)^T$ is a repelling equilibrium, $(1,0)^T$ is an equilibrium of saddle type. The Jacobian at the equilibria $(\pm\sqrt{1-a}, 1)^T$ has one negative eigenvalue (-1) and the other one is $1 \mp 2\sqrt{1-a} - \frac{a}{(1\pm\sqrt{1-a})^2}$ (it follows from the upper triangular form of the matrix that the eigenvalues are equal to the diagonal entries). Setting without loss of $a = 1 - b^2$ with $b \geq 0$, the latter eigenvalue is equal to

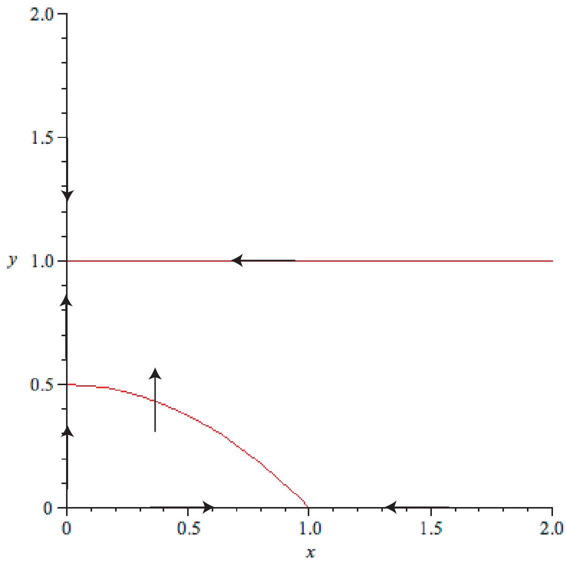
$$1 \mp 2b - \frac{1-b^2}{(1\pm b)^2} = \frac{1}{(1\pm b)^2}(1\pm b)^2(1\mp 2b) - (1-b^2) = \frac{1}{1\pm b}((1\pm b)(1\mp 2b) - (1\mp b)) = -\frac{1}{(1\pm b)}2b^2.$$

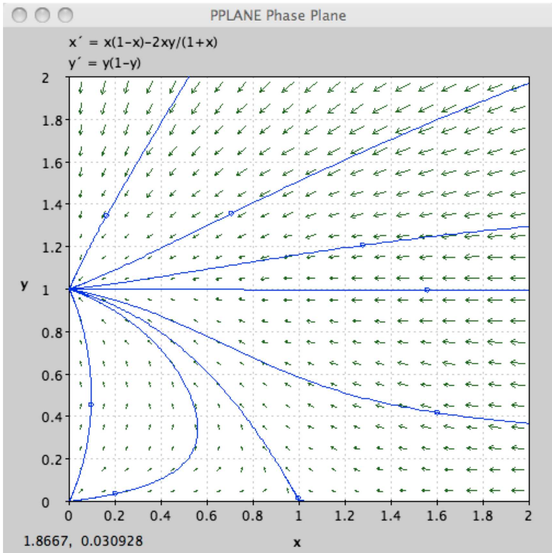
With $0 \leq b < 1$ we thus see that the second eigenvalue is negative if $a < 1$ $(b > 0)$ and 0 if $a = 1$ $(b = 0)$. Hence we conclude that the equilibria are attractors.

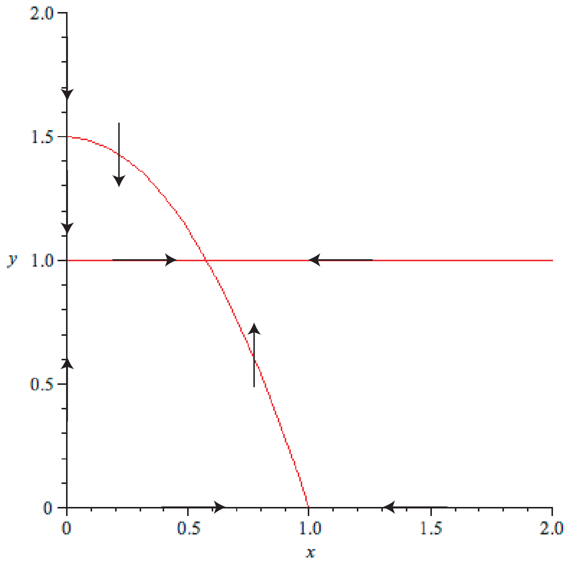
At $a = 1$ we have a bifurcation, where two equilibria branch off from another equilibrium, which itself persists, albeit with a change of stability. Such a bifurcation is sometimes known as a "pitchfork" bifurcation (after the corresponding bifurcation diagram that shows the bearmarks of a pitchfork).

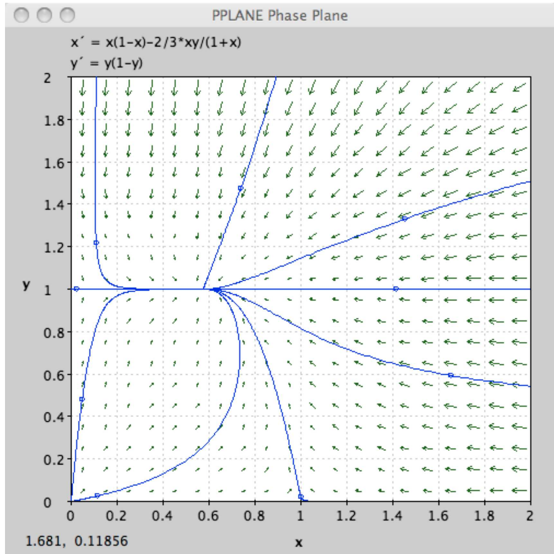
Bifurcation diagram: 

We sketch the nullclines and phase portraits for representatives values of a, $a = 2$ (before the pichfork bifurcation) and $a = 2/3$ (after the pitchfork bifurcation).

$a = 2$: 







$a = 2/3$:

From the nullclines we observe that all solutions converge to the stable (attracting) equilibrium; notice the "trapping regions" in the phase portrait for which we can see that any solution entering it stays inside it and converges to the stable equilibrium point.

3. [HSD, chap 11, q5] The nullclines are

$$\begin{aligned} \dot{x} = 0 \quad &: \quad x = 0 \ \text{ or } \ x + y = 2, \\ \dot{y} = 0 \quad &: \quad y = 0 \ \text{ or } \ 2x + y = 3. \end{aligned}$$

The equilibria (in the quadrant $x, y \geq 0$) are $(0,0)^T$, $(2,0)^T$, $(1,1)^T$ and $(0,3)^T$. Denoting the vector field as f, the Jacobian at $(x,y)^T$ is given by

$$Df\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 2 - 2x - y & -x \\ -2y & -2y + 3 - 2x \end{pmatrix},$$

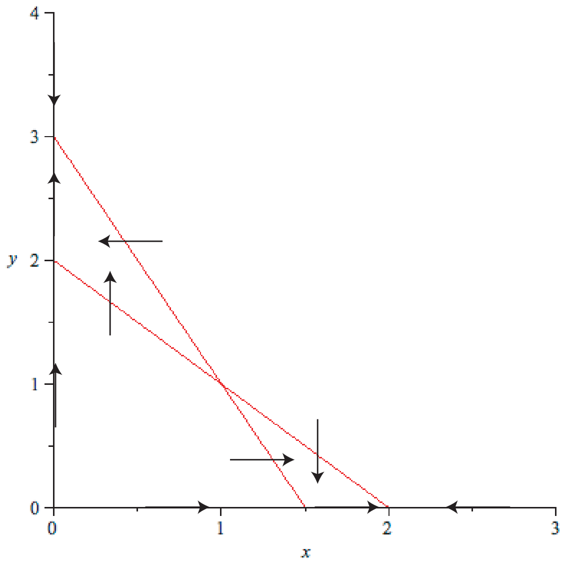
and in the equilibria

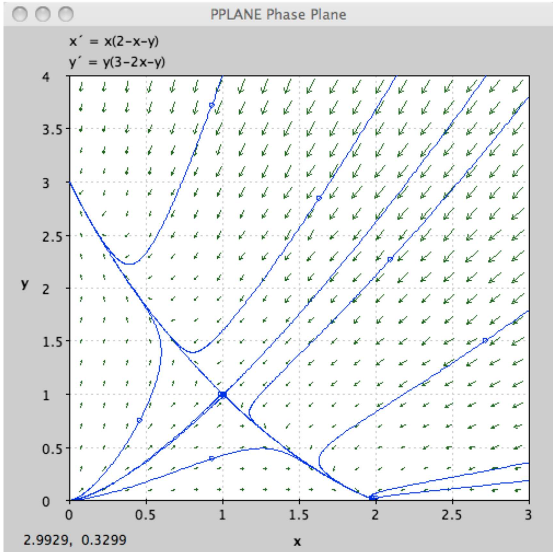
$$Df\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \quad Df\left(\begin{pmatrix} 2 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} -2 & -2 \\ 0 & -1 \end{pmatrix},$$

$$Df\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} -1 & -1 \\ -2 & -1 \end{pmatrix}, \quad Df\left(\begin{pmatrix} 0 \\ 3 \end{pmatrix}\right) = \begin{pmatrix} -1 & 0 \\ -6 & -3 \end{pmatrix}.$$

Hence the points $(0,3)^T$ and $(2,0)^T$ are attractors (asymptotically stable), $(0,0)^T$ is a repeller (asymptotically unstable) and the point $(1,1)^T$ is of saddle type (Jacobian has eigenvalues $1 \pm \sqrt{2}$).

We sketch the nullclines and phase portrait.





The equilibrium $(1,1)^T$ represents a situation where both species survive. However, it is very unlikely that a solution converges to this equilibrium as for this to occur the initial condition (and the entire orbit) must lie in the stable manifold of this saddle, which is a curve (on which most solutions do not lie). We note that the stable manifold of this saddle functions as the line that separates the basins of attraction for the two attracting equilibria.

4. [HSD, chap 11, q10] The nullclines are

$$\begin{array}{lll} \dot{x}=0 & : & x=0 \ \text{ or } \ y=\frac{1}{a}(1-x)(x+c), \\ \dot{y}=0 & : & y=0 \ \text{ or } \ x=y. \end{array}$$

The equilibria of this system are given by $(0,0)^T$, $(1,0)^T$, and $(z,z)^T$ with

$$z=\frac{1}{2}(1-(c+a)\pm\sqrt{1+2(c-a)+(c+a)^2}).$$

In order to establish the stability of the latter equilibrium, we first derive the general form of the Jacobian

$$Df\left(\begin{pmatrix} x \\ y \end{pmatrix}\right)=\begin{pmatrix} 1-2x-\frac{acy}{(x+c)^2} & -\frac{ax}{x+c} \\ by^2/x^2 & b-2y/x \end{pmatrix},$$

which with $x=y=z$ takes the form

$$Df\left(\begin{pmatrix} z \\ z \end{pmatrix}\right)=\begin{pmatrix} 1-2z-\frac{acz}{(z+c)^2} & -\frac{az}{z+c} \\ b & b-2 \end{pmatrix},$$

We now recall (not necessary, but I think elegant) that the eigenvalues $\lambda_\pm$ of a 2×2 matrix A have a simple expression in terms of its trace and determinant (if you do not already know this you can easily verify this by direct computation):

$$\lambda_\pm=\mathrm{tr}(A)/2\pm\sqrt{(\mathrm{tr}(A)/2)^2-\det(A)}.$$

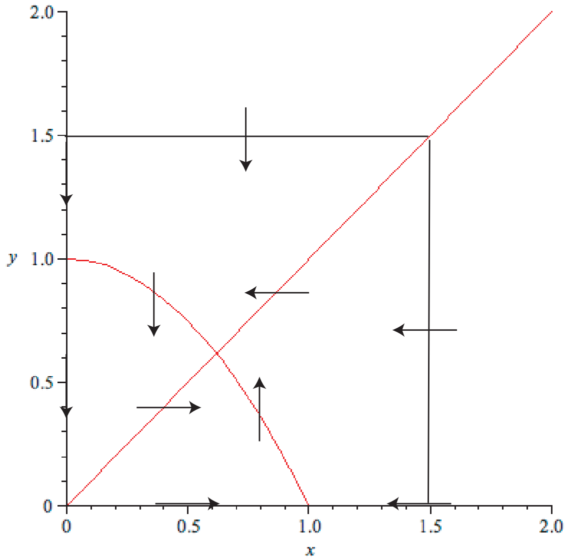
In particular, (asymptotic) stability requires that $Re(\lambda_\pm) < 0$, or equivalently that $\mathrm{tr}(A) < 0$ and $\det(A) > 0$. The conditions for stability are thus:

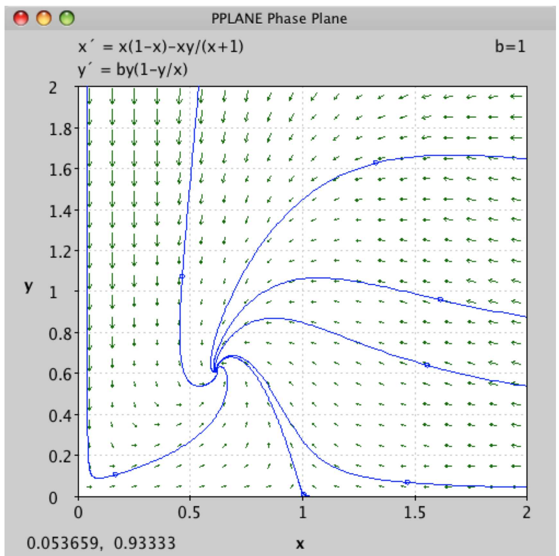
$$\begin{cases} z &= \frac{1}{2}(1-(c+a)+\sqrt{1+2(c-a)+(c+a)^2}) \\ 0 &> 1-2z-\frac{acz}{(z+c)^2}+b-2 \\ 0 &< (1-2z-\frac{acz}{(z+c)^2})(b-2)+\frac{abz}{z+c} \end{cases}$$

We have used here the fact that from geometric considerations, we know that the relevant root yielding z is the one obtained with the + sign. Consider the nullclines: the equilibrium point we consider lies on the intersection of the line $x = y$ and the quadratic $ay = (1-x)*(x+c)$, which intersect the x-axis at $x = 1$ and $x = -c < 0$. We are interested in the intersection that lies in the quadrant $x, y > 0$ which corresponds to the root with the + sign.

The question seems to suggest that there is a nice answer to the above set of equations, but I have not found one. I would be interested if any of you obtained some concise answer in terms of the parameters a, b, c. But using the explicit expression for z, one can find the boundaries of the regimes where the trace is negative and the determinant is positive by solve the expressions with equal sign for b. One obtains rather long formulas in the form of $b = f(a, c)$ and $b = g(a, c)$ for the surfaces where trace and determinant are equal to zero.

We can prove the existence of a limit cycle if this equilibrium is unstable, by application of the Poincaré-Bendixson theorem. The key is to observe that for all (positive) values of a, b and c we have a flow-invariant square box in the quadrant $x, y \geq 0$ that contains one equilibrium point. In the figure, I sketched the situation when $a = c = 1$, and the box $[0, 3/2] \times [0, 3/2]$ is forward flow-invariant. For illustrative purpose I also provided a phase portrait (in the case that $a = b = c = 1$),





This argument can be extended to hold for all parameter values, by choosing the flow-invariant box in the positive quadrant of appropriate size.

If the equilibrium is unstable, as a consequence of the Poincaré-Bendixson theorem (as discussed in the lecture) we obtain that there exists a (stable) periodic solution (that is part of the ω-limit set). We also know that this periodic solution must encircle the unstable equilibrium (because every periodic solution in the plane must encircle some equilibrium, see notes).

From 2008 exam question 4

4. (a) [5] $\dot{y} = 0$ if $x = 0$ or $y = 1 + x^2$. Then $\dot{x} = 0$ if $a = 0$ (but $a > 0$) and $a - 5x = 0$, respectively. Thus $(x, y) = (a/5, 1 + a^2/25)$ is the unique equilibrium. By the derivative test, asymptotic (in)stability is guaranteed if the real part of the eigenvalues of the Jacobian are all negative (positive).Its Jacobian is

$$\begin{pmatrix} -1 - 4\frac{y}{1+x^2} + 8x^2\frac{y}{(1+x^2)^2} & -4\frac{x}{1+x^2} \\ (1 - \frac{y}{1+x^2}) + 2x^2\frac{y}{(1+x^2)^2} & -\frac{x}{1+x^2} \end{pmatrix}$$

which evaluated at the equilibrium yields

$$J = \begin{pmatrix} \frac{-125+3a^2}{25+a^2} & -\frac{20a}{25+a^2} \\ \frac{2a^2}{25+a^2} & -\frac{5a}{25+a^2} \end{pmatrix}.$$

$\det J = 25a/(25 + a^2) > 0$ since $a > 0$. By the supplied formula this implies (since $|\sqrt{X^2 - Y}| < |X|$ if $Y > 0$) that the equilibrium cannot be of saddle type. So the sign of the real part of the eigenvalues is equal to the sign of the trace.$\mathrm{Tr}J = (3a^2 - 5a - 125)/(25 + a^2)$ is positive if $a > \frac{5}{6}(1 + \sqrt{61})$ and negative if $0 < a < \frac{5}{6}(1 + \sqrt{61})$.

(b) (i) [3] When $y = 0$, the vector field $(a - x, x)$ is transverse to the line $y = 0$ except when $x = 0$ (since $(a - x, x) \cdot (0, 1) = x$). When $x = 0$ the vector field $(a, 0)$ is transverse to the line $x = 0$. Hence on the x- and y-axis, the vector field points into the positive quadrant. Since solutions are tangent to the vector field, the flow thus maps the boundary of this quadrant into the interior of the quadrant. By continuity, hence the quadrant is flow invariant.

(ii) [4] At $x = c$ we have $\dot{x} = a - c - 4c\frac{y}{1+c^2}$. If $c > a$ and $0 \leq y \leq 1 + c^2$ then $\dot{x} < 0$. At $y = 1 + c^2$ we have $\dot{y} = x(1 - \frac{1+c^2}{1+x^2})$ so that $\dot{y} < 0$ if $x < c$. In combination of the result in part (a), it follows that on ∂B_c the vector field points into B_c so that $\Phi^t(\partial B_c) \subset \mathrm{int}(B_c)$ for all $t > 0$ and all $c \geq a$. Hence, no orbit of the flow can accumulate to any point outside B_a.

(iii) [4] The PBT asserts that if there is no equilibrium in the ω-limit set of a compact flow-invariant subset of the plane, then this subset must contain a periodic solution. This periodic solution bounds a compact simply connected flow-invariant subset of the plane which in turn must contain an equilibrium point (corollary from the PB theorem). As there is only one equilibrium every periodic solution must be encircling it.

(c) [4] When $a = \frac{5}{6}(1 + \sqrt{61})$, the eigenvalues of the Jacobian are purely imaginary. Still, typically, one would expect the equilibrium to be either asymptotically stable or unstable. If it was unstable, this would imply by the PBT that there existed a periodic solution. This periodic solution would typically be hyperbolic so that a continuation of it would exist for $a < \frac{5}{6}(1 + \sqrt{61})$ which contradicts the assumption. Hence we would expect the equilibrium to be asymptotically stable at the bifurcation point.