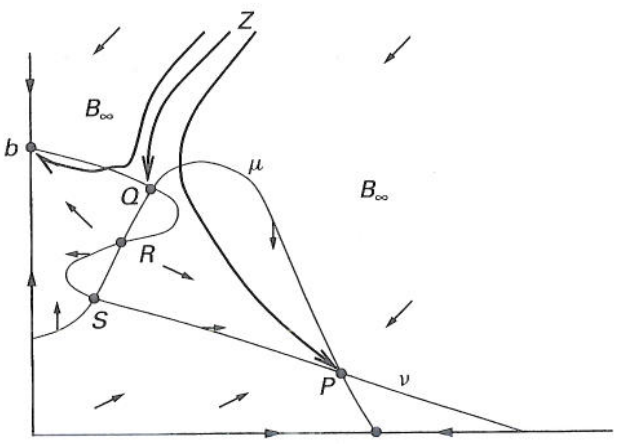


Figure 11.14 Note that solutions on either side of the point Z in the stable curve of Q have very different fates.

For example, this analysis tells us that, in Figure 11.14, only P and $(0, b)$ are asymptotically stable; all other equilibria are unstable. In particular, assuming that the equilibrium Q in Figure 11.14 is hyperbolic, then it must be a saddle because certain nearby solutions tend toward it, while others tend away. The point Z lies on one branch of the stable curve through Q. All points in the region denoted B_∞ to the left of Z tend to the equilibrium at $(0, b)$, while points to the right go to P. Thus as we move across the branch of the stable curve containing Z, the limiting behavior of solutions changes radically. Since solutions just to the right of Z tend to the equilibrium point P, it follows that the populations in this case tend to stabilize. On the other hand, just to the left of Z, solutions tend to an equilibrium point where $x = 0$. Thus in this case, one of the species becomes extinct. A small change in initial conditions has led to a dramatic change in the fate of populations. Ecologically, this small change could have been caused by the introduction of a new pesticide, the importation of additional members of one of the species, a forest fire, or the like. Mathematically, this event is a jump from the basin of P to that of $(0, b)$.

11.4 Exploration: Competition and Harvesting

In this exploration we will investigate the competitive species model where we allow either harvesting (emigration) or immigration of one of the species. We

consider the system

$$x' = x(1 - ax - y)$$
$$y' = y(b - x - y) + h.$$

Here a, b, and h are parameters. We assume that $a, b > 0$. If $h < 0$, then we are harvesting species y at a constant rate, whereas if $h > 0$, we add to the population y at a constant rate. The goal is to understand this system completely for all possible values of these parameters. As usual, we only consider the regime where $x, y \geq 0$. If $y(t) < 0$ for any $t > 0$, then we consider this species to have become extinct.

1. First assume that $h = 0$. Give a complete synopsis of the behavior of this system by plotting the different behaviors you find in the a, b parameter plane.
2. Identify the points or curves in the ab–plane where bifurcations occur when $h = 0$ and describe them.
3. Now let $h < 0$. Describe the ab–parameter plane for various (fixed) h-values.
4. Repeat the previous exploration for $h > 0$.
5. Describe the full three-dimensional parameter space using pictures, flip books, 3D models, movies, or whatever you find most appropriate.

EXERCISES

1. For the SIRS model, prove that all solutions in the triangular region Δ tend to the equilibrium point $(\tau, 0)$ when the total population does not exceed the threshold level for the disease.

2. Sketch the phase plane for the following variant of the predator/prey system:

$$x' = x(1 - x) - xy$$
$$y' = y\left(1 - \frac{y}{x}\right).$$

3. A modification of the predator/prey equations is given by

$$x' = x(1 - x) - \frac{axy}{x + 1}$$
$$y' = y(1 - y)$$

where $a > 0$ is a parameter.

(a) Find all equilibrium points and classify them.
(b) Sketch the nullclines and the phase portraits for different values of a.
(c) Describe any bifurcations that occur as a varies.

4. Another modification of the predator/prey equations is given by

$$x' = x(1-x) - \frac{xy}{x+b}$$
$$y' = y(1-y)$$

where $b > 0$ is a parameter.

(a) Find all equilibrium points and classify them.
(b) Sketch the nullclines and the phase portraits for different values of b.
(c) Describe any bifurcations that occur as b varies.

5. The equations

$$x' = x(2-x-y),$$
$$y' = y(3-2x-y)$$

satisfy conditions (1) through (3) in Section 11.3 for competing species. Determine the phase portrait for this system. Explain why these equations make it mathematically possible, but extremely unlikely, for both species to survive.

6. Consider the competing species model

$$x' = x(a-x-ay)$$
$$y' = y(b-bx-y)$$

where the parameters a and b are positive.

(a) Find all equilibrium points for this system and determine their stability type. These types will, of course, depend on a and b.
(b) Use the nullclines to determine the various phase portraits that arise for different choices of a and b.
(c) Determine the values of a and b for which there is a bifurcation in this system and describe the bifurcation that occurs.
(d) Record your findings by drawing a picture of the ab–plane and indicating in each open region of this plane the qualitative structure of the corresponding phase portraits.

7. Two species x, y are in *symbiosis* if an increase of either population leads to an increase in the growth rate of the other. Thus we assume

$$x' = M(x,y)x$$
$$y' = N(x,y)y$$

with

$$\frac{\partial M}{\partial y} > 0 \quad \text{and} \quad \frac{\partial N}{\partial x} > 0$$

and $x, y \geq 0$. We also suppose that the total food supply is limited; hence for some $A > 0, B > 0$ we have

$$M(x,y) < 0 \quad \text{if } x > A,$$
$$N(x,y) < 0 \quad \text{if } y > B.$$

If both populations are very small, they both increase; hence

$$M(0,0) > 0 \quad \text{and} \quad N(0,0) > 0.$$

Assuming that the intersections of the curves $M^{-1}(0), N^{-1}(0)$ are finite, and that all are transverse, show the following:

(a) Every solution tends to an equilibrium in the region $0 < x < A$, $0 < y < B$.
(b) There are no sources.
(c) There is at least one sink.
(d) If $\partial M/\partial x < 0$ and $\partial N/\partial y < 0$, there is a unique sink Z, and Z is the ω-limit set for all (x,y) with $x > 0, y > 0$.

8. Give a system of differential equations for a pair of mutually destructive species. Then prove that, under plausible hypotheses, two mutually destructive species cannot coexist in the long run.

9. Let y and x denote predator and prey populations. Let

$$x' = M(x,y)x$$
$$y' = N(x,y)y$$

where M and N satisfy the following conditions.

(a) If there are not enough prey, the predators decrease. Hence for some $b > 0$

$$N(x,y) < 0 \quad \text{if } x < b.$$

(b) An increase in the prey improves the predator growth rate; hence $\partial N/\partial x > 0$.

(c) In the absence of predators a small prey population will increase; hence $M(0,0) > 0$.

(d) Beyond a certain size, the prey population must decrease; hence there exists $A > 0$ with $M(x,y) < 0$ if $x > A$.

(e) Any increase in predators decreases the rate of growth of prey; hence $\partial M/\partial y < 0$.

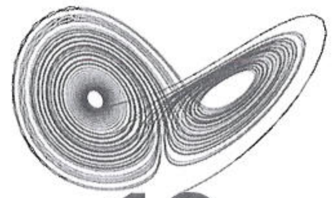
(f) The two curves $M^{-1}(0), N^{-1}(0)$ intersect transversely and at only a finite number of points.

Show that if there is some (u,v) with $M(u,v) > 0$ and $N(u,v) > 0$ then there is either an asymptotically stable equilibrium or an ω-limit cycle. Moreover, show that, if the number of limit cycles is finite and positive, one of them must have orbits spiraling toward it from both sides.

10. Consider the following modification of the predator/prey equations:

$$x' = x(1-x) - \frac{axy}{x+c}$$
$$y' = by\left(1 - \frac{y}{x}\right)$$

where $a, b,$ and c are positive constants. Determine the region in the parameter space for which this system has a stable equilibrium with both $x, y \neq 0$. Prove that, if the equilibrium point is unstable, this system has a stable limit cycle.



12 Applications in Circuit Theory

In this chapter we first present a simple but very basic example of an electrical circuit and then derive the differential equations governing this circuit. Certain special cases of these equations are analyzed using the techniques developed in Chapters 8 through 10 in the next two sections; these are the classical equations of Lienard and van der Pol. In particular, the van der Pol equation could perhaps be regarded as one of the fundamental examples of a nonlinear ordinary differential equation. It possesses an oscillation or periodic solution that is a periodic attractor. Every nontrivial solution tends to this periodic solution; no linear system has this property. Whereas asymptotically stable equilibria sometimes imply death in a system, attracting oscillators imply life. We give an example in Section 12.4 of a continuous transition from one such situation to the other.

12.1 An *RLC* Circuit

In this section, we present our first example of an electrical circuit. This circuit is the simple but fundamental series *RLC* circuit displayed in Figure 12.1. We begin by explaining what this diagram means in mathematical terms. The circuit has three *branches*, one resistor marked by R, one inductor marked by L, and one capacitor marked by C. We think of a branch of this circuit as a from 2008 exam

4. Consider the following model of the chemical reaction between two substances whose concentrations are denoted by x and y, respectively:

$$\begin{aligned}\frac{dx}{dt} &= a - x - \frac{4xy}{1+x^2},\\ \frac{dy}{dt} &= x\left(1 - \frac{y}{1+x^2}\right).\end{aligned}$$

Here a is a positive parameter. Note also that as x and y represent concentrations, we are only interested in $x, y \geq 0$. The model serves to illustrate that chemical reactions may yield asymptotic solutions that oscillate instead of being stationary.

(a) (i) Show that there is a unique equilibrium and that at this equilibrium the derivative of the vector field (Jacobian) is equal to

$$\frac{1}{25+a^2}\begin{pmatrix} -125+3a^2 & -20a \\ 2a^2 & -5a \end{pmatrix}.$$

(ii) Show that the equilibrium is asymptotically stable if $a < \frac{5}{6}(1+\sqrt{61})$ and asymptotically unstable if $a > \frac{5}{6}(1+\sqrt{61})$.
[You may apply the *derivative test* without proof. Hint: Recall that the eigenvalues of a 2×2 matrix A are given by $\lambda_{\pm} = \mathrm{tr}(A)/2 \pm \sqrt{(\mathrm{tr}(A)/2)^2 - \det(A)}$, where $\mathrm{tr}(A)$ denotes the trace of A and $\det(A)$ its determinant.]

(b) Show that

(i) The quadrant $\{(x,y) \mid x \geq 0,\ \mathbf{y} \geq 0\}$ is positive flow-invariant.

(ii) All ω-limit sets of the flow are contained in the region

$$B_a := \{(x,y) \mid a \geq x \geq 0,\ 1+a^2 \geq y \geq 0\}.$$

[Hint: consider the flow through the boundary of B_c for all $c \geq a$.]

(iii) Apply the Poincaré-Bendixson Theorem to show that there exists a periodic solution in B_a if $a > \frac{5}{6}(1+\sqrt{61})$, and that this periodic solution must encircle the equilibrium.

(c) Suppose that the equilibrium is the unique ω-limit set of the ODE when $a < \frac{5}{6}(1+\sqrt{61})$. What stability property would you expect for the equilibrium at $a = \frac{5}{6}(1+\sqrt{61})$? Motivate your answer.