M2AA1 Differential Equations
Exercise sheet 6 answers

1. We write $f(\mathbf{x},\lambda)$ with $\mathbf{x}=(x,y)\in\mathbb{R}^2$. It is also easier to treat $f(\mathbf{x},\lambda)\in\mathbb{R}^2$ as a column vector. Then the Jacobian for $\mathbf{x}=0$ at $\lambda=0$ is given by $D_1f(0,0)=\begin{pmatrix}2&1\\2&1\end{pmatrix}$, so that we have $\ker D_1f(0,0)=\left\langle\begin{pmatrix}1\\-2\end{pmatrix}\right\rangle$ and $\mathrm{range}D_1f(0,0)=\left\langle\begin{pmatrix}1\\1\end{pmatrix}\right\rangle$. We may choose (for instance) the complementary spaces as $C=\tilde{C}=\left\langle\begin{pmatrix}1\\0\end{pmatrix}\right\rangle$ other choices will lead to different details in the computations, but of course in the end to the same overall result). In order to find f_1 and f_2 we determine the projections P_1 and P_2 so that for all $\mathbf{x}\in\mathbb{R}^2$ we have $\mathbf{x}=P_1\mathbf{x}+P_2\mathbf{x}$ with $P_1\mathbf{x}\in\mathrm{range}D_1f(0,0)$ and $P_2\in\tilde{C}$. P_1 is defined by the fact that

$$P_1\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix}\quad\text{and}\quad P_1\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\quad\Rightarrow P_1=\begin{pmatrix}0&1\\0&1\end{pmatrix},\quad P_2=I-P_1=\begin{pmatrix}1&-1\\0&0\end{pmatrix}.$$

We define $f_1=P_1\circ f$ and $f_2=P_2\circ f$ to obtain

$$f(\mathbf{x},\lambda)=0\quad\Leftrightarrow\quad\begin{cases}f_1(\mathbf{x},\lambda)=2x+(1+\lambda)y-xy=0,\\ f_2(\mathbf{x},\lambda)=\lambda-x^2-\lambda y+xy=0.\end{cases}$$

(It should be noted that I here suppressed the fact that of course the formulas for f_1 and f_2 should be formally multiplied by the relevant basis vectors of $\mathrm{range}D_1f(0,0)$ and $\tilde{C}$, respectively. But of course the zeros of f correspond to zeros of the system made up of f_1 and f_2.) We are now left to write $\mathbf{x}\in\mathbb{R}^2$ as a sum of vectors $P_3\mathbf{x}\in\ker D_1f(0,0)$ and $P_4\mathbf{x}\in C$. It follows that

$$P_3=\begin{pmatrix}0&-\frac{1}{2}\\0&1\end{pmatrix},\quad P_4=I-P_3=\begin{pmatrix}1&\frac{1}{2}\\0&0\end{pmatrix}\quad\Rightarrow\quad\begin{pmatrix}x\\y\end{pmatrix}=-\frac{y}{2}\begin{pmatrix}1\\-2\end{pmatrix}+(x+\frac{y}{2})\begin{pmatrix}1\\0\end{pmatrix}.$$

So with new coordinates u,v with respect to the basis vector of $\ker D_1f(0,0)$ and C, i.e. $(x,y)^T=u(1,-2)^T+v(1,0)^T$ we have $y=-2u$ and $x=v+u$. These leads us, in terms of this new coordinates to the following expressions

$$\begin{aligned}&f_1:\ker D_1f(0,0)\times C\times\mathbb{R}\to\mathrm{range}D_1f(0,0), && f_1(u,v,\lambda)=2v-2\lambda u+2uv+2u^2\\ &f_2:\ker D_1f(0,0)\times C\times\mathbb{R}\to\tilde{C}, && f_2(u,v,\lambda)=\lambda+2\lambda u-3u^2-4uv-v^2.\end{aligned}$$

It is readily verified that $D_2f_1(0,0,0)=2\neq0$ and $D_1f_2(0,0,0)=0$ as guaranteed by the construction of f_1 and f_2. We find

$$f_1(u,v,\lambda)=0\quad\Leftrightarrow\quad v(u,\lambda)=(\lambda u-u^2)/(1+u),$$

yielding

$$g(u,\lambda):=f_2(u,v(u),\lambda)=\lambda+2\lambda u-3u^2-\frac{4u(\lambda u-u^2)}{1+u}-\left(\frac{\lambda u-u^2}{1+u}\right)^2.$$

As $\frac{\partial}{\partial\lambda}g(0,0)=1\neq0$ by application of the IFT we have a unique $\lambda(u)$ so that $g(u,\lambda(u))=0$. From the expression of g we obtain that $\lambda(u)$ near $u=0$ is given by

$$\lambda(u)=3u^2+O(u^3).$$

Hence the solution curve in (x,y,λ)-coordinates has up to second order in u the form

$$(x,y,z)=(u+v(u,\lambda),-2u,\lambda(u))=(u-u^2,-2u,3u^2)+O(u^3),$$

since $v(u)=(1-u)(\lambda u-u^2)+O(u^3)=\lambda u-(\lambda+1)u^2+O(u^3)=-u^2+O(u^3)$. And one verifies indeed that for this curve we have $f(\mathbf{x},\lambda)=O(u^3)$, consistent with the accuracy of our approximation to the solution.

2. (a) Given the proposed decompositions, we use the projections P to range$Df(0)$ (with ker$P = \tilde{C}$) and $(I-P)$ to $\tilde{C}$ to define $f_1 := P \circ f$ and $f_2 : (I-P) \circ f$.

   (b) By linearity of $Df(0)$ we have $\dim C = \dim \text{range}Df(0)$. Since if there were $\mathbf{x}, \mathbf{y} \in C$ with $\mathbf{x} \neq \mathbf{y}$ such that $Df(0)\mathbf{x} = Df(0)\mathbf{y}$, then $\mathbf{x} - \mathbf{y} \in \ker Df(0)$ which contradicts the fact that C is complementary to ker$Df(0)$. Hence, $Df(0)|_C : C \to \text{range}Df(0)$ is a linear onto map between two isomorphic finite dimensional vector spaces (i.e. with the same dimension), and hence invertible. We finally show that $Df(0)_C = D_2f(0,0)$. Consider $Df(0) : \ker Df(0) \times C \to \text{range}Df(0) \times \tilde{C}$, then in matrix form we have

$$Df(0) = \begin{pmatrix} D_1f_1(0,0) & D_2f_1(0,0) \\ D_1f_2(0,0) & D_2f_1(0,0) \end{pmatrix},$$

   with all entries denoting matrices, in general. It the follows by construction that for all $\mathbf{y} \in C$ $Df(0)\mathbf{y} = D_2f_1(0,0)\mathbf{y} + D_2f_2(0,0)\mathbf{y} \in \text{range}Df(0)$, which in turn implies that $D_2f_2(0,0)\mathbf{y} = 0$ since range$D_2f_2(0,0) \subset \tilde{C}$. Hence $D_2f_1(0,0) = Df(0)|_C$, and thus invertible. Similarly, $Df(0)|_{\ker Df(0)} = 0$ and for all $\mathbf{x} \in \ker Df(0)$ we have $0 = Df(0)\mathbf{x} = D_1f_1(0,0)\mathbf{x} + D_1f_2(0,0)\mathbf{x} \in \tilde{C}$ implying that $0 = D_1f_2(0,0)\mathbf{x}$ (apply the projection $(I-P)$ to both sides.)

3. If $L = -I$, $V = |\mathbf{x}|$ is a Lyapunov function, since $dV/dt = -|\mathbf{x}|$.

4. Note that the 2nd order ODE can be written (or thought of) as a system of two coupled first order ODEs.

   (a) $E = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\omega^2 x^2$ and $\frac{d}{dt}E = \dot{x}\ddot{x} + \omega^2 x\dot{x} = 0$. $x = \dot{x} = 0$ is an absolute minimum of E. So $x = \dot{x} = 0$ is a Lyapunov stable equilibrium.

   (b) Try again $E = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\omega^2 x^2$. Then $\frac{d}{dt}E = \dot{x}\ddot{x} + \omega^2 x\dot{x} = -\varepsilon\dot{x}^2$ so $\frac{d}{dt}E < 0$ unless $\dot{x} = 0$ in which case $\frac{d}{dt}E = 0$. From this it follows that $x = \dot{x} = 0$ is a Lyapunov stable equilibrium.
   If the equilibrium is not asymptotically stable, some orbit must not converge to the equilibrium. From the proof of the theorem about Lyapunov functions we know that such an orbit must accumulate to a solution of the ODE for which $\frac{d}{dt}E = 0$ (along the solution). But we find that on the line $\dot{x} = 0$ the vector field contains no other solutions that the equilibrium $x = \dot{x} = 0$ since whenever $x \neq 0$ the vector field is always transverse to the line $\dot{x} = 0$ (so all solution curves through these points are also transverse to this line in these points). The parameter $\varepsilon > 0$ can have the interpretation of a friction coefficient (damping the velocity).

   (c) We now choose, in analogy to before, $E = \frac{1}{2}|\dot{\mathbf{x}}|^2 + \frac{1}{2}\omega^2|\mathbf{x}|^2$ and find that $\frac{d}{dt}E = \dot{\mathbf{x}} \cdot \ddot{\mathbf{x}} + \omega^2 \mathbf{x} \cdot \dot{\mathbf{x}} = -\varepsilon|\dot{\mathbf{x}}|^2$, and obtain the same conclusion as in (b).

5. This equation admits Lyapunov function $V(x,y,z) = x^2 + 2y^2 + z^2$, namely

$$dV/dt = x\dot{x} + 2y\dot{y} + z\dot{z} = 2xy(z-1) - 2yx(z-1) - z^2 = -z^2 \leq 0.$$

   Thus the origin is Lyapunov stable. On the other hand consider the solution $\mathbf{x}(t)$ with $\mathbf{x}(0) = (0, \varepsilon, 0)$ is

$$\mathbf{x}(t) = (-\sqrt{2}\sin(\sqrt{2}t)\varepsilon, \cos(\sqrt{2}t)\varepsilon, 0).$$

   Obviously this solution does not tend to the origin as $t \to \infty$. Hence the origin is not asymptotically stable. Note that the plane $z = 0$ on which $\frac{d}{dt}V = 0$ is flow invariant.

6. V is a (strict) Lyapunov function for the ODE with time running backwards, yielding $\mathbf{x}_0$ to be asymptotically stable for this ODE. Hence, for the ODE with time running forward the equilibrium is not Lyapunov stable. (Maybe some more details could be given to explain the latter implication.)

7. (a) Suppose that $\frac{d}{dt}V(\mathbf{y}(t)) \neq 0$. Then, since $\frac{d}{dt}V \leq 0$ along all orbits, for some T sufficiently large we have $V(\mathbf{y}(t)) < C\ \forall t > T$. This contradicts the fact that the solution $\mathbf{x}(t)$ accumulates $\mathbf{y}_0$, since by continuity of the flow the solution $\mathbf{x}(t)$ must accumulate to every point in the positive semi-orbit of the flow emanating from $\mathbf{y}_0$.

(b) The orbit $\mathbf{x}(t)$ may accumulate to a closed loop consisting of an equilibrium point and a so-called *homoclinic* orbit that converges to this equilibrium as $t \to \pm\infty$.

8. In the lecture we defined $\omega(\mathbf{x})$ as the set of accumulation points of the positive semi-orbit of the flow Φ^t emanating from $\mathbf{x}$, i.e. for all $\mathbf{y} \in \omega(\mathbf{x})$ there exists a monotone increasing sequence t_k with $k \in \mathbb{N}$ so that $\lim_{k\to\infty} t_k = \infty$ and $\lim_{k\to\infty} \Phi^{t_k}(\mathbf{x}) = \mathbf{y}$. For notational convenience, let use refer to the ω-limit set defined in this way as $\omega(\mathbf{x})$ and let the set defined in the question be defined as $\omega'(\mathbf{x})$. The aim is to show that $\omega(\mathbf{x}) = \omega'(\mathbf{x})$.

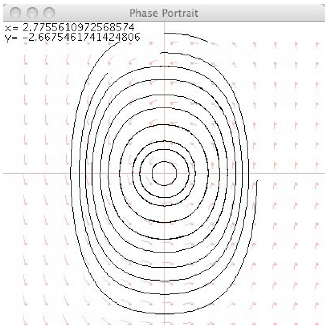
   $\omega(\mathbf{x}) \subset \omega'(\mathbf{x})$: suppose $\lim_{k\to\infty} \Phi^{t_k}(\mathbf{x}) = \mathbf{y}$ then clearly $\mathbf{y} \in \overline{\{\Phi^t(\mathbf{x}) \mid t > T\}}$ for all $T \in \mathbb{R}^+$ and thus also $\mathbf{y} \in \omega'(\mathbf{x})$.

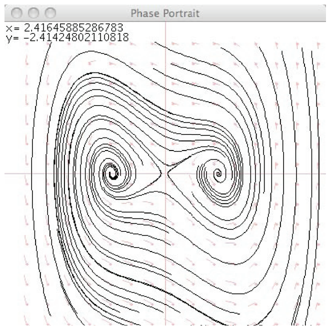
   $\omega'(\mathbf{x}) \subset \omega(\mathbf{x})$: if $\mathbf{y} \in \omega'(\mathbf{x})$ we have that $\mathbf{y} \in \overline{\{\Phi^t(\mathbf{x}) \mid t > T\}}$ for all $T \in \mathbb{R}^+$. In turn this means that there must be an increasing sequence t_k such that $\Phi^{t_k}(\mathbf{x}) \to \mathbf{y}$ since if not then there would be a T sufficiently large so that $\mathbf{y} \notin \overline{\{\Phi^t(\mathbf{x}) \mid t > T\}}$.

9.
   - We saw in Question 5 of Sheet 5 that $r = 1$ is a globally attracting circle (with basin of attraction $\mathbb{R}^2 \setminus \{0\}$), so the ω-limits sets must be contained in this circle or the origin $r = 0$. As the circle is a periodic solution, the entire circle $r = 1$ is the only ω-limit set of the system. This is circle is also the α-limit set of all points on the circle $r = 1$. The origin $r = 0$ is an equilibrium and its own α- and ω-limit set.
   - Again here the flow in the angular direction is constant. In the radial direction the flow has "equilibria" at $r = 0, 1, 2$. The derivative of the radial vector field at these points is $2, -1, 2$ respectively, so the origin $r = 0$ is an unstable equilibrium, the circle $r = 1$ an asymptotically stable periodic solution and the circle $r = 2$ an asymptotically unstable periodic solution. The equilibria and periodic solutions are always their own α- and ω-limit sets. The periodic solution $r = 1$ also functions as the ω-limit set of the region $0 < r < 2$ (its basin of attraction).
   - flow in angular direction is again constant. Invariant circles (periodic solutions) arise at $r = 0 \bmod \pi$. Derivatives in radial direction are $\cos(r)$, i.e. $r = 0$ is unstable equilbrium, $r = 1$ stable per soln, $r = 2$ unstable per soln etc. This equilibrium and periodic solutions form the set of all α- and ω-limit sets of this flow.
   - Equilbria are the points $(x, y) = (0 \bmod \pi, \pi/2 \bmod \pi)$ and $(x, y) = (\pi/2 \bmod \pi, 0 \bmod \pi)$. It turns out that these equilbria are the only ω- and α limit sets of this vector field.

     (Note: This is an example of a special type of vector field, called a *gradient* vector field. Let $F(x, y) := \cos x \sin y$, then
     $$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = -\nabla F(x, y) = -\begin{pmatrix} \frac{\partial F(x,y)}{\partial x} \\ \frac{\partial F(x,y)}{\partial y} \end{pmatrix}.$$
     It turns out that gradient vector fields $f(\mathbf{x}) = -\nabla F(\mathbf{x})$ have natural Lyapunov functions, namely the function F:
     $$\frac{d}{dt}F(\mathbf{x}) = \nabla F(\mathbf{x}) \cdot \frac{d}{dt}\mathbf{x} = -(\nabla F(\mathbf{x})) \cdot (\nabla F(\mathbf{x})) = -|\nabla F(\mathbf{x})|^2 \leq 0.$$
     As a consequence, we have that $x \in \omega(y)$ implies that $\frac{d}{dt}F(\mathbf{x}) = 0$ which in turn implies that $\nabla F(\mathbf{x}) = 0$ which implies that $f(\mathbf{x}) = 0$ so that $\mathbf{x}$ must be an equilibrium point.)

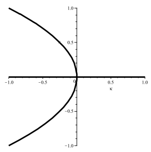
10. (a) $\dot{x} = y$, $\dot{y} = -\kappa x - x^3 - \mu y$.
    (b) (i) $\frac{dE}{dt} = y\dot{y} + y(\kappa x + x^3) = -\mu y^2 \leq 0$
        (ii) Sketches of the phase portraits at $(\kappa, \mu) = (1, 0)$ and $(\kappa, \mu) = (-1, 0.5)$. [Anything with main features in common will do.]





If $\kappa > 0$ and $\mu = 0$ the set of α- and ω-limit sets consist of the entire $\mathbb{R}^2$ since all solutions are periodic or equilibria. If $\kappa < 0$ and $\mu > 0$ the α- and ω- limit sets are the three equilibria.

(c) (i) Equilibria are determined by $\kappa x + x^3 = 0 \Leftrightarrow x = 0$ or $x = \pm\sqrt{-\kappa}$.



(ii) With the added term, we find the equlbria by solving $\kappa x + x^3 - \varepsilon x^7 = 0$, giving $x = 0$ or $0 = \kappa + x^2 - \varepsilon x^6 =: h(x^2, \varepsilon)$. Write $y = x^2$. Then we have $h(-\kappa, 0) = 0$ and the partial derivative with respect to the first argument $\partial_1 h(-\kappa, 0) \neq 0$. Then by the Implicit Function Theorem we have a unique $y(\kappa, \varepsilon)$ with $y(\kappa, 0) = -\kappa$ such that $h(y(\varepsilon), \varepsilon) = 0$. As $y(\varepsilon)$ depends continuously (and smoothly) on ε the qualitative shape of the bifurcation diagram will be preserved.

(d) With $f(x, y) = (f_1(x, y), f_2(x, y))$, the nullclines are defined as the sets (curves) $f_1(x, y) = 0$ and $f_2(x, y) = 0$.

The Jacobian $Df(x, y)$ has row vectors $(\frac{\partial f_j(x,y)}{\partial x}, \frac{\partial f_j(x,y)}{\partial y})$ $j = 1, 2$. Invertibility of $Df(x, y)$ means that these row vectors are linearly independent.

The nulclines intersect transversely in (x_0, y_0) if $f_1(x_0, y_0) = f_2(x_0, y_0) = 0$ and the tangent vectors to these curves in (x_0, y_0) are linearly independent. These tangent vectors are orthogonal to the normals $\nabla f_1(x_0, y_0)$ and $\nabla f_1(x_0, y_0)$. The tangent vectors are linearly independent if and only if the vectors normal to them are linearly independent. We now note that the gradient vectors representing the normals are precisely (the transpose of) the row vectors that arise in the Jacobian above. Hence invertibility of $Df(x_0, y_0)$ is equivalent to transverse intersections of nullclines.

By the implicit function theorem, we now that an equilibrium (x_0, y_0) is persistent (under small perturbations) if $Df(x_0, y_0)$ is invertible. We also know that transverse intersections are persistent. So the above result is consistent with these results on persistence.