M2AA1 Differential Equations
Exercise sheet 6

1. Use Lyapunov-Schmidt reduction to find an expression (or approximation) of the set of equilibria (as a function of external parameter $\lambda \in \mathbb{R}$) of the planar vector field $f(x, y, \lambda) = (\lambda+2x+y-x^2, 2x+(1+\lambda)y-xy)$ near the equilibrium $(x, y) = (0, 0)$ at $\lambda = 0$, where $(x, y) \in \mathbb{R}^2$.

2. Let $f : \mathbb{R}^m \to \mathbb{R}^m$ and $Df(0)$ denote its derivative at 0. Consider the following decompositions of $\mathbb{R}^m$:
$$\mathbb{R}^m = \mathrm{ker}Df(0) \oplus C, \quad \mathbb{R}^m = \mathrm{range}Df(0) \oplus \tilde{C},$$
where (thus) C and $\tilde{C}$ are complementary to $\mathrm{ker}Df(0)$ and $\mathrm{range}Df(0)$, respectively, in $\mathbb{R}^m$.
   (a) Show that by choosing appropriate projections (to $\mathrm{range}Df(0)$ and $\tilde{C}$, respectively) f can be written as the system
$$f_1 : \mathrm{ker}Df(0) \times C \to \mathrm{range}Df(0), \quad f_2 : \mathrm{ker}Df(0) \times C \to \tilde{C}.$$
   (b) Show that $D_2 f_1(0, 0)$ is invertible, and that $D_1 f_2(0, 0) = 0$. (As usual the notation D_j means "derivative with respect to the jth argument".)

3. Find a Lyapunov function for the linear ODE $\dot{\mathbf{x}} = -\mathbf{x}$, with $\mathbf{x} \in \mathbb{R}^m$.

4. (a) Consider the one-dimensional harmonic oscillator $\ddot{x} = -\omega^2 x$ with $\omega > 0$. Show that the total energy of this oscillator (recall your mechanics!) is a Lyapunov function for the equilibrium point $x = \dot{x} = 0$, so that the latter is Lyapunov stable.
   (b) Consider the slight modifcation $\ddot{x} = -\omega^2 x - \varepsilon\dot{x}$, where $\varepsilon > 0$. Find a Lyapunov function to prove that the equilibrium $x = \dot{x} = 0$ is asymptotically stable and that its basin of attraction consists of the entire phase space. (We say that this equilibrium is a global attractor.) Give a physical interpretation to the parameter ε.
   (c) Find a Lyapunov function to prove that the equilibrium $\mathbf{x} = \dot{\mathbf{x}} = 0$ is a global attractor of $\ddot{\mathbf{x}} = -\omega^2\mathbf{x}-\varepsilon\dot{\mathbf{x}}$, were $\omega, \varepsilon > 0$ and $\mathbf{x} \in \mathbb{R}^m$.

5. Consider the ODE
$$\dot{x} = 2y(z - 1), \dot{y} = -x(z - 1), \dot{z} = -z.$$
Note that $(x, y, z) = (0, 0, 0)$ is an equilibrium solution.
   (a) Show that this equilibrium is Lyapunov stable by finding a Lyapunov function (Hint: there exists a Lyapunov function that is a quadratic polynomial).
   (b) Is this equilibrium also asymptotically stable? Why (not)?

6. Suppose that V is a smooth function defined in a neighbourhood U of an equilibrium point $\mathbf{x}_0$ of the ODE $\dot{\mathbf{x}} = f(\mathbf{x})$, such that $V(\mathbf{x}_0) = 0$, $V(\mathbf{x}) > 0$ for all $\mathbf{x} \in U \setminus \{\mathbf{x}_0\}$, and $\frac{d}{dt}V(\mathbf{x}(t)) > 0$, where $\mathbf{x}(t) \in U \setminus \{\mathbf{x}_0\}$ and $\mathbf{x}(t)$ satisfies the ODE. Prove that as a consequence, $\mathbf{x}_0$ is not Lyapunov stable.

7. Consider a solution $\mathbf{x}(t)$ with initial condition $\mathbf{x}_0 = \mathbf{x}(0)$ of an the ODE $\dot{x} = f(\mathbf{x})$, with $\mathbf{x} \in \mathbb{R}^m$ and Lyapunov function V, i.e.$\frac{d}{dt}V(\mathbf{x}(t)) \leq 0$. Suppose that $\mathbf{x}(t)$ accumulates to a point $\mathbf{y}_0$.
   (a) Show that the solution $\mathbf{y}(t)$ with $\mathbf{y}(0) = \mathbf{y}_0$ satisfies $\frac{d}{dt}V(\mathbf{y}(t)) = 0$ for all $t \geq 0$.
   (b) Show that the orbit $Y := \{\mathbf{y}(t) \mid t \in \mathbb{R}\}$ is contained in the ω-limit set of $\mathbf{x}_0$, i.e. $Y \subset \omega(\mathbf{x}_0)$, but that the orbit $\mathbf{x}(t)$ does not necessary converge to Y as $t \to \infty$. (In the sense that $\lim_{t\to\infty} \inf_{\mathbf{y}\in Y} |\mathbf{x}(t) - \mathbf{y}|$ does not necessarily converge to 0.) [For instance, sketch an example.]

8. Show that the ω-limit set of a point $\mathbf{x} \in \mathbb{R}^m$ for a flow $\Phi^t : \mathbb{R}^m \to \mathbb{R}^m$ is given by
$$\omega(\mathbf{x}) = \bigcap_{T\in\mathbb{R}^+} \overline{\{\Phi^t(\mathbf{x}) \mid t > T\}}.$$
(In other words show that this definition is equivalent to the one given in the lecture.)

9. For each of the following systems, identify all points that lie in either an α- or ω-limit set ((r,θ) denote polar coordinates)

- $\dot{r} = r - r^2, \quad \dot{\theta} = 1$
- $\dot{r} = r^3 - 3r^2 + 2r, \quad \dot{\theta} = 1$
- $\dot{r} = \sin r, \quad \dot{\theta} = -1$
- $\dot{x} = \sin x \sin y, \quad \dot{y} = -\cos x \cos y$

10. (from 2008 exam) Consider the equations of motion for a nonlinear oscillator with friction

$$\frac{d^2x}{dt^2} = -\kappa x - x^3 - \mu \frac{dx}{dt}, \tag{0.1}$$

where $x \in \mathbb{R}$, μ is a non-negative parameter (friction constant) and κ is a constant that can be both positive and negative (elasticity constant).

(a) Write the equation (0.1) as a first order ODE on the plane $\mathbb{R}^2$.

(b) (i) Show that the *energy* of this oscillator

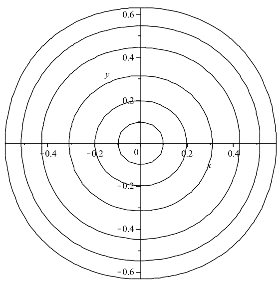
$$E = \frac{1}{2}\left(\frac{dx}{dt}\right)^2 + \frac{\kappa}{2}x^2 + \frac{1}{4}x^4$$

is a Lyapunov function of (0.1).

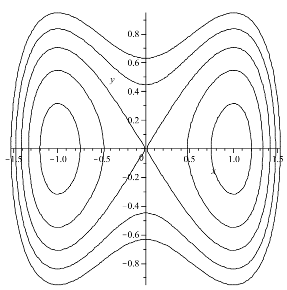
(ii) Sketch the phase portrait and describe all ω- and α limit sets of this system in the case that:

* $\kappa = 1$ and $\mu = 0$.
* $\kappa = -1$ and $\mu > 0$.

You may use the following sketches of the contours (E =constant) of the Lyapunov function E when $\kappa = 1$ and $\kappa = -1$:



$\kappa = 1$



$\kappa = -1$

(c) (i) Analyze the bifurcation of equilibria in the system (0.1) as κ increases through $\kappa = 0$. Sketch the bifurcation diagram (bifurcation parameter κ versus x).

(ii) Discuss whether (and if, how) the bifurcation diagram in (ii) changes if one adds a term εx^7 (with $|\varepsilon|$ being very small) to the right-hand-side of (0.1).

(d) Equilibria can be viewed as the intersection of *nullclines* of the planar vector field derived in (a). Recall that the nullclines are defined as curves on which one of the components of the vector field is equal to zero.

Show for general planar vector fields $f : \mathbb{R}^2 \to \mathbb{R}^2$, that the derivative of the vector field $Df(\mathbf{x}_0)$ at an equilibrium $\mathbf{x}_0$ is invertible if and only if the nullclines have a transverse intersection at $\mathbf{x}_0$.

Discuss this relationship in the context of conditions for persistence of equilibria under small perturbations.