M2AA1 Differential Equations
Exercise sheet 5 answers

1. (a) If $|B| \leq \delta$ then all the matrix elements b_{ij} of B satisfy $|b_{ij}| \leq \delta$. (This can be verified from the fact that if $b := \max b_{ij}$ then there exists a vector $\mathbf{x}$ with $|\mathbf{x}| = 1$ so that $|B\mathbf{x}| \geq b$.) Now the eigenvalues λ of $A + B$ are determined as the roots of the equation $f(\lambda) := \det(A + B - \lambda I) = 0$. $f : \mathbb{C} \to \mathbb{C}$ is polynomial and the roots $f^{-1}(0)$ depend continuously on the elements of $A+B$. But if $|B| \leq \delta$ then by the above argument we also have $|(A + B)_{ij} - a_{ij}| = |b_{ij}| \leq \delta$. Hence, by continuity of the eigenvalues as a function of the matrix elements, if A is hyperbolic (and thus all eigenvalues μ of A satisfy the condition that $|\mathrm{Re}(\mu)| \geq \varepsilon$) then there exists a $\delta > 0$ such that for all B with $|B| \leq \delta$ each eigenvalue λ_i of $A + B$ lies in an open ball of radius $\varepsilon/2$ around an eigenvalue μ_i of A such that in the limit of $|B| \to 0$ we have $\lambda_i \to \mu_i$.

   (b) We noticed in (a) already that $|B| < \delta$ implies that $|b_{ij}| < \delta$. On the other hand, if $|b_{ij}| < \delta/m$ for all $i, j = 1, \ldots, m$ it follows that $|B| < \delta$. An (open) ε-ball around A is defined as all $A + B$ with $d(A + B, A) < \varepsilon$, i.e. $\sqrt{\sum_{i,j=1}^{m} b_{ij}^2} < \varepsilon$. I thus follows that the set $\{A + B \mid |B| < \delta\} \supset \{A + B \mid d(A + B, A) < \delta/m\}$ and thus that it is a neighbourhood of A.

2. equilibria, Jacobian J

   - $(x, y) = (0 \bmod \pi, \frac{\pi}{2} \bmod \pi)$, $J = \begin{pmatrix} \cos(x) & 0 \\ 0 & -\sin(y) \end{pmatrix}$. In equilibria J has eigenvalues ± 1, ie hyperbolic.
   - $(x, y) = (0, 0)$; J at this point is 0 matrix, ie not hyperbolic
   - $(x, y) = (0, 0)$; $J = \begin{pmatrix} 1 & 2y \\ 0 & 2 \end{pmatrix}$ has evals $1, 2$ so hyperbolic asymptotically unstable equilibrium.
   - $(x, y) = (0, 0)$; $J = \begin{pmatrix} 0 & 2y \\ 1 & 0 \end{pmatrix}$ is non-hyperbolic in equilbrium.
   - $(x, y) = (0, 0)$ and $J = 0$ in equilbrium: non-hyperbolic.

   In the non-hyperbolic cases one finds that the local flow is not well-predicted by the linear approximation (in doubt you can check your phase portraits with maple/matlab etc).

3. Try $u = x + ay^2$ and $v = z + by^2$ then we have $\dot{u} = u$ and $\dot{v} = -v$ provided that $a = 1/3$ and $b = 1$.
$$\dot{u} = \dot{x} + \frac{2}{3}\dot{y}y = x + y^2 - \frac{2}{3}y^2 = u, \quad \dot{v} = \dot{z} + 2\dot{y}y = -z + y^2 - 2y^2 = -v.$$
   Hence $(\dot{u}, \dot{y}, \dot{v}) = (u, -y, -z)$.

4. Equilibria satisfy $y = -x^2$ and $y = x + a$, hence $x^2 + x + a = 0 \Leftrightarrow x = \frac{1}{2}(-1 \pm \sqrt{1 - 4a})$. Jacobian $J(x, y) = \begin{pmatrix} 2x & 1 \\ 1 & -1 \end{pmatrix}$. The eigenvalues of J are $\lambda_\pm = -1 + \frac{\varepsilon}{2}\sqrt{1 - 4a} \pm \frac{1}{2}\sqrt{5 - 4a}$ for $x_\varepsilon = \frac{1}{2}(-1 + \varepsilon\sqrt{1 - 4a})$ and $\varepsilon = \pm 1$. At $a = \frac{1}{4}$ we have a fold bifurcation of equilibria (none exist when $a > \frac{1}{4}$ and two when $a = \frac{1}{4}$). Correspondingly, eigenvalues of the Jacobian at $a = \frac{1}{4}$ are $\lambda_\pm = -1 \pm 1 = \{0, -2\}$, when $a < \frac{1}{4}$ then x_1 is hyperbolic of saddle type and x_{-1} is a hyperbolic (asymptotically stable) attractor.

5. In the radial direction, when $r = 1$ we have $\dot{r} = 0$. Moreover the derivative of the vector field $\dot{r} = f(r) = r - r^2$ in the radial direction is equal to $\frac{df(r)}{dr} = 1 - 2r$, which is negative (equal to -1) if $r = 1$. Hence the circle $r = 1$ is globally attracting (note that $f(r) < 0$ if $r > 1$ and $f(r) > 0$ if $r < 1$. We also have an equilbrium $r = 0$ that is asymptotically unstable $\frac{df}{dr}(0) = 1$.

   The vector field for the θ-component give insight about the flow on the invariant circle $r = 1$. Note that $0 \leq \sin^2(\theta/2) \leq 1$. Hence we have 2 equilibria when $0 < a < 1$, one equilibrium when $a = 0$ or $a = 1$ and no equilbria if $a < 0$ or $a > 1$ (in which case the invariant circle constitutes a periodic solution). The bifurcations at $a = 0$ and $a = 1$ are fold bifurcations. At $a = 0$ we have one equilibrium $\theta = 0$ that splits into one stable ($\theta < 0$) and one unstable ($\theta > 0$) equilbrium $\theta = 2\arcsin(\pm\sqrt{a})$ that move apart until the reunit at the other side of the circle $\theta = \pi$ when $a = 1$. There these equilibria recombine and disappear.