M2AA1 Differential Equations
Exercise sheet 5

1. (a) Prove that if $A \in gl(m, \mathbb{R})$ is hyperbolic, there exists a $\delta > 0$ such that $A + B$ is also hyperbolic for all $B \in gl(m, \mathbb{R})$ with $|B| \leq \delta$.

   (b) Argue that $\{A + B \mid A, B \in gl(m, \mathbb{R}),\ |B| < \delta\}$ is a neighbourhood of A in $gl(m, \mathbb{R})$ if we consider $gl(m, \mathbb{R}) \simeq \mathbb{R}^{m^2}$ as a metric space with the metric induced from the natural Euclidean metric on $\mathbb{R}^{m^2}$: $d(A, B) = \sqrt{\sum_{i,j=1}^{m} (a_{ij} - b_{ij})^2}$, where a_{ij} and b_{ij} denote the matrix elements of A and B.

2. For each of the following systems:

   - $\dot{x} = \sin x, \quad \dot{y} = \cos y$
   - $\dot{x} = x(x^2 + y^2), \quad \dot{y} = y(x^2 + y^2)$
   - $\dot{x} = x + y^2, \quad \dot{y} = 2y$
   - $\dot{x} = y^2, \quad \dot{y} = x$
   - $\dot{x} = x^2, \quad \dot{y} = y^2$

   (a) Find all equilibria and describe the behaviour of the associated linearized system.

   (b) Sketch the phase portrait for the nonlinear system. Does the linearized systems accurately describe the local behaviour near the equilibrium points?

3. Find a global change of coordinates that linearizes the system

$$\begin{cases} \dot{x} &= x + y^2 \\ \dot{y} &= -y \\ \dot{z} &= -z + y^2 \end{cases}$$

4. Consider the system

$$\begin{cases} \dot{x} &= x^2 + y \\ \dot{y} &= x - y + a \end{cases}$$

   where a is a parameter.

   (a) Find all equilibria and describe the behaviour of the linearized system at each of them.

   (b) Describe any bifurcations that occur when a is varied.

5. Discuss the local and global behaviour of solutions of the system

$$\begin{cases} \dot{r} &= r - r^2 \\ \dot{\theta} &= \sin^2(\theta/2) - a \end{cases}$$

   where a is a parameter and (r, θ) are polar coordinates for $\mathbb{R}^2$. In particular determine for which values of a bifurcations occur.