M2AA1 Differential Equations
Exercise sheet 4 answers

(from Chap 17, Hirsch, Smale and Devaney pp 404-405)

1. (a) exact solution $x(t) = 2 - e^t$. Picard iterates: $u_0(t) = 1$, $u_1(t) = 1-t$, $u_2(t) = 1-t-\frac{1}{2}t^2$ etc. Corresponds to Taylor expansion of exact soln.
   (b) exact solution $x(t) = 0$. Picard iterates: $u_0(t) = 0$, $u_1(t) = 0$, $u_2(0) = 0$ etc.
   (c) exact solution $x(t) = (\frac{-3}{t-3})^3$. Picard iterates: $u_0(t) = 1$, $u_1(t) = 1+t$, $u_2(t) = \frac{4}{7} + \frac{3}{7}(1+t)^{7/3} = 1 + t + \frac{2}{3}t^2 + o(t^2)$ etc. Develops same Taylor series expansion.
   (d) exact $t = \ln(\sec(x) + \tan(x))$ (for those of you who are good at doing these special integrals; do not worry if this is not something you see quickly). Picard iterates: $u_0(t) = 0$, $u_1(t) = t$, $u_2(t) = \sin(t)$, etc
   (e) exact $x(T) = \exp(t/2 - 1/2)$. Picard iterates: $u_0(t) = 1$, $u_1(t) = 1/2 + t/2$, $u_2(t) = 5/8 + t/4 + t^2/8$ etc

2. by induction: we have $u_0 = x_0$; if $u_n(t) = \sum_{k=0}^{n} \frac{t^k A^k}{k!} x_0$ then
$$u_{n+1}(t) = x_0 + \int_0^t A u_n(s) ds = x_0 + \sum_{k=0}^{n} \frac{t^{n+1}A^{n+1}}{(n+1)!} x_0 = \sum_{k=0}^{n+1} \frac{t^k A^k}{k!} x_0.$$

3. $\dot{x} = y$, $\dot{y} = -x$ with $(x, y) = \mathbf{x}$ we have $\dot{\mathbf{x}} = A\mathbf{x}$ where $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. We note that $A^2 = -I$ hence
$$\exp(At) = \sum_{k=0}^{\infty} (\frac{A^{2k}t^{2k}}{(2k)!} + \frac{A^{2k+1}t^{2k+1}}{(2k+1)!}) = \sum_{k=0}^{\infty} ((-1)^k \frac{t^{2k}}{(2k)!} I + \frac{(-1)^k t^{2k+1}}{(2k+1)!} A)$$
and
$$\begin{pmatrix} \cos(t) \\ -\sin(t) \end{pmatrix} = \exp(At) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k}}{(2k)!} \\ \sum_{k=0}^{\infty} -\frac{(-1)^k t^{2k+1}}{(2k+1)!} \end{pmatrix}.$$

4. Lipschitz constants K: (a) $K = 1$ (triangle inequality); (b) not Lipschitz because derivative is unbounded; (c) $K = 1$ (bounded derivative); (d) $K = 1$ (derivative bounded by largest eigenvalue); (e) Lipschitz since bounded derivative (gradient).

5. $x(t) = 0$ and $x(t) = (2t/3)^{3/2}$.

7. $x(t) = t + 1$ $t < 0$ and $x(t) = 2t + 1$ for $t \geq 0$ (not differentiable). if $f'(x) = 0$, $x > 1$ the solution for $t < 0$ remains identical and $x(t) = 1$ for $t \geq 0$.

10. solution $x(t) = a\cos(t) + b\sin(t)$, then $x(0) = 0$ implies $a = 0$ but then also $x(\pi) = 0$.

11. solution $x(t) = a\cos(t\sqrt{k}) + b\sin(t\sqrt{k})$, then $x(0) = 0$ implies $a = 0$ and $x(\pi) = b\sin(\pi\sqrt{k}) = 1$ can be solved as long as $\sqrt{k} \notin \mathbb{Z}$.

12. Let $U(t) = C + \int_0^t u(s)v(s)ds$ with $C > 0$, then $u(t) \leq U(t)$. Since $U'(t) = u(t)v(t)$ we have $U'(t)/U(t) \leq v(t)$ from which we obtain $\frac{d}{dt} \ln U(t) \leq v(t)$ and hence $u(t) \leq U(t) \leq \exp(\int_0^t v(s)ds)$. Since true for all $C > 0$ by continuity also true for $C = 0$.