Assume now that $|\xi| \leq \delta_1$. From the previous equation, we find, for $t \in J$,

$$g(t) \leq N \int_0^t g(s)\,ds + \int_0^t \epsilon|\xi|e^{Ks}\,ds,$$

so that

$$g(t) \leq N \int_0^t g(s)\,ds + C\epsilon|\xi|$$

for some constant C depending only on K and the length of J. Applying Gronwall's inequality we obtain

$$g(t) \leq C\epsilon e^{Nt}|\xi|$$

if $t \in J$ and $|\xi| \leq \delta_1$. (Recall that δ_1 depends on ϵ.) Since ϵ is any positive number, this shows that $g(t)/|\xi| \to 0$ uniformly in $t \in J$, which proves the proposition.

EXERCISES

1. Write out the first few terms of the Picard iteration scheme for each of the following initial value problems. Where possible, use any method to find explicit solutions. Discuss the domain of the solution.
   (a) $x' = x - 2; x(0) = 1$
   (b) $x' = x^{4/3}; x(0) = 0$
   (c) $x' = x^{4/3}; x(0) = 1$
   (d) $x' = \cos x; x(0) = 0$
   (e) $x' = 1/2x; x(1) = 1$
2. Let A be an $n \times n$ matrix. Show that the Picard method for solving $X' = AX, X(0) = X_0$ gives the solution $\exp(tA)X_0$.
3. Derive the Taylor series for $\cos t$ by applying the Picard method to the first-order system corresponding to the second-order initial value problem

$$x'' = -x; \quad x(0) = 1, \quad x'(0) = 0.$$

4. For each of the following functions, find a Lipschitz constant on the region indicated, or prove there is none:
   (a) $f(x) = |x|, -\infty < x < \infty$
   (b) $f(x) = x^{1/3}, -1 \leq x \leq 1$

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   (c) $f(x) = 1/x, 1 \leq x \leq \infty$
   (d) $f(x,y) = (x+2y, -y), (x,y) \in \mathbb{R}^2$
   (e) $f(x,y) = \dfrac{xy}{1+x^2+y^2}, x^2+y^2 \leq 4$
5. Consider the differential equation

$$x' = x^{1/3}.$$

How many different solutions satisfy $x(0) = 0$?

6. What can be said about solutions of the differential equation $x' = x/t$?
7. Define $f: \mathbb{R} \to \mathbb{R}$ by $f(x) = 1$ if $x \leq 1$; $f(x) = 2$ if $x > 1$. What can be said about solutions of $x' = f(x)$ satisfying $x(0) = 1$, where the right-hand side of the differential equation is discontinuous? What happens if you have instead $f(x) = 0$ if $x > 1$?
8. Let $A(t)$ be a continuous family of $n \times n$ matrices and let $P(t)$ be the matrix solution to the initial value problem $P' = A(t)P, P(0) = P_0$. Show that

$$\det P(t) = (\det P_0)\exp\left(\int_0^t \operatorname{Tr} A(s)\,ds\right).$$

9. Suppose F is a gradient vector field. Show that $|DF_X|$ is the magnitude of the largest eigenvalue of DF_X. (*Hint*: DF_X is a symmetric matrix.)
10. Show that there is no solution to the second-order two-point boundary value problem

$$x'' = -x, \quad x(0) = 0, \ x(\pi) = 1.$$

11. What happens if you replace the differential equation in the previous exercise by $x'' = -kx$ with $k > 0$?
12. Prove the following general fact (see also Section 17.3): If $C \geq 0$ and $u, v: [0, \beta] \to \mathbb{R}$ are continuous and nonnegative, and

$$u(t) \leq C + \int_0^t u(s)v(s)\,ds$$

for all $t \in [0, \beta]$, then $u(t) \leq Ce^{V(t)}$, where

$$V(t) = \int_0^t v(s)\,ds.$$

13. Suppose $C \subset \mathbb{R}^n$ is compact and $f: C \to \mathbb{R}$ is continuous. Prove that f is bounded on C and that f attains its maximum value at some point in C.