M2AA1 Differential Equations
Exercise sheet 3

1. Show that from the conditions on the distance function d (in the definition of a metric space) it follows that the distance is *positive definite*: $d(x, y) \geq 0$ for all $x, y \in X$.

2. Decide, with proof, which of the following are complete metric spaces (with the natural metric): $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$, $[0, 1]$, and $[0, 1)$.

3. Suppose that X is a bounded and closed subset of $\mathbb{R}^n$, and $F : X \to X$ is *shrinking* such that

$$d(F(x), F(y)) < d(x, y), \text{ for any } x \neq y.$$

   Prove that F has a unique fixed point $x_0 \in X$ and $\lim_{n\to\infty} F^n(x) = x_0$ for all $x \in X$. Can you give an example where the convergence is not exponential?

4. Prove that (as appears at the end of Inverse Function Theorem on $\mathbb{R}$), if $F : \mathbb{R} \to \mathbb{R}$ is C^1 (continous with continuous first derivative) and F is invertible near x_0, then $(F^{-1})'(y) = 1/F'(x)$ for $y = F(x)$ near $F(x_0)$.

5. Consider an equilibrium $\mathbf{x}_0$ of an autonomous ODE $\dot{\mathbf{x}} = f(\mathbf{x})$ in $\mathbb{R}^m$. Use the derivative test in $\mathbb{R}^m$ (see course notes) to show that if all eigenvalues of the derivative $Df(\mathbf{x}_0)$ have negative real part, then the flow near the equilibrium is a contraction and hence that the equilibrium is asymptotically stable. (Note that this generalizes an earlier similar observation for equilibria of *linear* ODEs.) [Hint: show first that the derivative of the time-t flow at $\mathbf{x}_0$ is given by $D\Phi^t(\mathbf{x}_0) = \exp(Df(\mathbf{x}_0)t)$.]

6. (a) Consider the *polar coordinate transformation*

$$\varphi : \mathbb{R}^2 \to \mathbb{R}^2 \quad \text{where} \quad \varphi(r, \theta) = (r\cos\theta, r\sin\theta).$$

   Show that φ is *locally invertible* whenever $r \neq 0$. Is φ invertible on $\mathbb{R}^2 \setminus \{r = 0\}$?

   (b) Along the same lines, describe the invertibility properties of the *spherical coordinate transformation*

$$\Sigma : \mathbb{R}^3 \to \mathbb{R}^3 \quad \text{where} \quad \Sigma(r, \theta, \phi) = (r\sin\phi\cos\theta, r\sin\phi\sin\theta, r\cos\theta).$$

7. For the following maps $f : \mathbb{R}^m \to \mathbb{R}^m$ and points $\mathbf{a} \in \mathbb{R}^m$

   (i) $m = 2$, $f(x, y) = (\cos x, xy)$, $\mathbf{a} = (\pi, -1)$

   (ii) $m = 3$, $f(x, y, z) = (3x + y, z - 3y, x + z)$, $\mathbf{a} = (2, -3, 5)$

   (iii) $m = 3$, $f(x, y, z) = (xy, yz, xz)$, $\mathbf{a} = (1, 0, -1)$

   (iv) $m = 3$, $f(x, y, z) = (xe^y, xyz, \ln|z|)$, $\mathbf{a} = (2, 0, 1)$

   establish whether f is invertible in a neighbourhood of the point $\mathbf{a}$. If so, where possible find an explicit expression for the inverse of otherwise find its Taylor expansion up to the lowest nonlinear order (that has a nontrivial contribution). In that case also calculate the derivative of f^{-1} at $f(\mathbf{a})$.

8. Suppose that two smooth surfaces of dimension n and m in $\mathbb{R}^k$ $(n, m < k)$ intersect each other in a point $\mathbf{p}$. Let us assume that (locally, near $\mathbf{p}$) the surfaces are defined by the maps $F : \mathbb{R}^n \to \mathbb{R}^k$ and $G : \mathbb{R}^m \to \mathbb{R}^k$, respectively. Note that a point of intersection of the two surfaces is a root of the map $H : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^k$ defined by $H(\mathbf{x}, \mathbf{y}) = F(\mathbf{x}) - G(\mathbf{y})$, i.e. if p is a point of intersection then there exists $\mathbf{x}$ (coordinates on the n-dimensional surface) and $\mathbf{y}$ (coordinates on the m-dimensional surface) such that $p = F(\mathbf{x}) = G(\mathbf{y})$. Answer the following questions under the assumption that $k < m + n$. If you have difficulty working with general k, m, n you are advised to first work out answers in the special cases that $k = 3$ with $n = m = 2$ (intersection of two 2-dimensional surfaces in $\mathbb{R}^3$) and $n = m + 1 = 2$ (intersection of a 1-dimensional curve and a 2-dimensional surface in $\mathbb{R}^3$).

   (a) Suppose that F and G are linear maps, and $p = 0$ is the point of intersection. Identify a condition on F and G that implies that the dimension of the intersection of the surfaces is equal to $m + n - k$.

   (b) Formulate a corresponding condition involving the derivatives of the maps F and G in the intersection point so that the dimension of the intersection is also equal to $m + n - k$ if the maps F and G are not linear. (This should generalize an observation made in the lecture (and course notes) for the intersection of two curves in $\mathbb{R}^2$.)

   (c) The condition you have identified in (b) is called *transversality*. Show that such *transverse* intersections of smooth surfaces are persistent (under smooth perturbations).

9. Prove the Inverse Function Theorem in $\mathbb{R}^m$, as stated in the course notes.