M2AA1 Differential Equations
Exercise sheet 2 answers

1. (a) Let $(\mathbf{x}, y) \in \mathbb{R}^m \times \mathbb{R}$ and rewrite the ODE as the system $\frac{d\mathbf{x}}{dt} = f(\mathbf{x}, y)$ and $\frac{dy}{dt} = 1$.

   (b) We have $\mathbf{x}(t) = \Phi^{t,t_0}(\mathbf{x}(t_0))$ and $\frac{d\mathbf{x}(t)}{dt} = f(\mathbf{x}, t)$. By definition, we have

$$\begin{aligned}\frac{d\mathbf{x}}{dt}(t_0) &= \lim_{\varepsilon\to 0} \frac{\mathbf{x}(t_0+\varepsilon) - \mathbf{x}(t_0)}{\varepsilon}, \\ &= \lim_{\varepsilon\to 0} \frac{\Phi^{t_0+\varepsilon,t_0}(\mathbf{x}(t_0)) - \Phi^{t_0,t_0}(\mathbf{x}(t_0))}{\varepsilon}, \\ &= \frac{d}{dt}\Phi^{t,t_0}|_{t=t_0}(\mathbf{x}(t_0)).\end{aligned}$$

   Hence $f(\mathbf{x}, t) = \frac{d}{ds}\Phi^{s,t}|_{s=t}(\mathbf{x})$.

   (c) We first write the ODE as

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -t \\ t \end{pmatrix}.$$

   By a linear change of coordinates (to a basis consisting of the eigenvectors of the matrix $\begin{pmatrix} 2 & 1 \\ 2 & -1 \end{pmatrix}$) the two coupled nonautonomous equations decouple. Let

$$\begin{pmatrix} x \\ y \end{pmatrix} = P\begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} 1 & -\frac{3-\sqrt{17}}{4} \\ 1 & -\frac{3+\sqrt{17}}{4} \end{pmatrix}\begin{pmatrix} v \\ w \end{pmatrix}$$

   then

$$\frac{d}{dt}\begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} \frac{1+\sqrt{17}}{2} & 0 \\ 0 & \frac{1-\sqrt{17}}{2} \end{pmatrix}\begin{pmatrix} v \\ w \end{pmatrix} + \begin{pmatrix} \frac{7-\sqrt{17}}{4}t \\ \frac{7+\sqrt{17}}{4}t \end{pmatrix}.$$

   The decoupled equations are of the form $\dot{z} = az + bt$. For each of these the initial value problem $z(t_0) = z$ can be solved as $z(t) = -bt/a - b/a^2 + e^{at}c$, where $c = e^{-t_0 a}(z + bt_0/q + b/q^2)$. The associated flow Φ^{t_1,t_0} of this non-autonomous systems has the form

$$\Phi^{t_1,t_0}(z) = -bt_1/a - b/a^2 + e^{a(t_1-t_0)}(z + bt_0/a + b/a^2).$$

   (Note that we can verify that the relation obtained in (b) indeed holds.)
   The flow for the original equation is obtained by transforming coordinates back to $(x, y)^T$: in the $(v, w)^T$ coordinates

$$\begin{aligned}\Phi^{t_1,t_0}\begin{pmatrix} v \\ w \end{pmatrix} =& \begin{pmatrix} e^{\frac{1+\sqrt{17}}{2}(t_1-t_0)} & 0 \\ 0 & e^{\frac{1-\sqrt{17}}{2}(t_1-t_0)} \end{pmatrix}\begin{pmatrix} v \\ w \end{pmatrix} + \\ & \begin{pmatrix} \frac{-7+\sqrt{17}}{2(1+\sqrt{17})}t_1 - \frac{(-7+\sqrt{17})^2}{1+\sqrt{17}} + e^{\frac{1+\sqrt{17}}{2}(t_1-t_0)}(-\frac{-7+\sqrt{17}}{2(1+\sqrt{17})}t_0 + \frac{(-7+\sqrt{17})^2}{1+\sqrt{17}}) \\ \frac{-7-\sqrt{17}}{2(1-\sqrt{17})}t_1 - \frac{(-7-\sqrt{17})^2}{1-\sqrt{17}} + e^{\frac{1-\sqrt{17}}{2}(t_1-t_0)}(-\frac{-7-\sqrt{17}}{2(1-\sqrt{17})}t_0 + \frac{(-7-\sqrt{17})^2}{1-\sqrt{17}}) \end{pmatrix}.\end{aligned}$$

   and in $(x, y)^T$ coordinates

$$\begin{aligned}\Phi^{t_1,t_0}\begin{pmatrix} x \\ y \end{pmatrix} =& P\begin{pmatrix} e^{\frac{1+\sqrt{17}}{2}(t_1-t_0)} & 0 \\ 0 & e^{\frac{1-\sqrt{17}}{2}(t_1-t_0)} \end{pmatrix}P^{-1}\begin{pmatrix} x \\ y \end{pmatrix} + \\ & P\begin{pmatrix} \frac{-7+\sqrt{17}}{2(1+\sqrt{17})}t_1 - \frac{(-7+\sqrt{17})^2}{1+\sqrt{17}} + e^{\frac{1+\sqrt{17}}{2}(t_1-t_0)}(-\frac{-7+\sqrt{17}}{2(1+\sqrt{17})}t_0 + \frac{(-7+\sqrt{17})^2}{1+\sqrt{17}}) \\ \frac{-7-\sqrt{17}}{2(1-\sqrt{17})}t_1 - \frac{(-7-\sqrt{17})^2}{1-\sqrt{17}} + e^{\frac{1-\sqrt{17}}{2}(t_1-t_0)}(-\frac{-7-\sqrt{17}}{2(1-\sqrt{17})}t_0 + \frac{(-7-\sqrt{17})^2}{1-\sqrt{17}}) \end{pmatrix}.\end{aligned}$$

2. (a) We treat the matrix first as (a special case of) a complex matrix. There are two eigenvalues $\lambda_\pm = \alpha \pm i\beta$. Let $\mathbf{v}_\pm$ be the eigenvectors for $\lambda_\pm$. Because A is a real matrix it follows that $\mathbf{v}_- = \overline{\mathbf{v}}_+$. We seek a basis for E_{λ_-}. Since the eigenvalue is complex, for the moment we will treat the problem as if it was posed in $\mathbb{C}^4$ and we will return later to $\mathbb{R}^4$. We thus seek a basis for E_{λ_-} (as complex vector space). Since λ_- has algebraic multiplicity 2, it is generated by 2 linearly independent vectors (in $\mathbb{C}^4$). One natural candidate for a basis vector is the eigenvector $\mathbf{v}_-$. But since $\dim E_{\lambda_-} = 2$, we need another linearly independent one. We call this $\mathbf{w}_-$. We know that $\mathbf{w}_-$ is not an eigenvector of A, because if it were, it would be of the form $c\mathbf{v}_-$ for some $c \in \mathbb{C}$, and thus not linearly independent of $\mathbf{v}_-$.

   We repeat this procedure for E_{λ_+}. Having already made a choice for $\mathbf{w}_-$, we choose the basis for E_{λ_+} as $\mathbf{v}_+$ and $\mathbf{w}_+ = \overline{\mathbf{w}}_-$. It is not difficult to verify that if $\mathbf{v}_-$ and $\mathbf{w}_-$ are linearly independent, then so are $\mathbf{w}_-$ and $\mathbf{w}_+$.

   So we have $E_{\lambda_-} = \mathrm{span}_{\mathbb{C}}(\mathbf{v}_-, \mathbf{w}_-)$ and $E_{\lambda_+} = \mathrm{span}_{\mathbb{C}}(\mathbf{v}_+, \mathbf{w}_+)$.

   We know that $A\mathbf{v}_\pm = \lambda_\pm \mathbf{v}_\pm$. We now consider the images $A\mathbf{w}_\pm$. We know about $\mathbf{w}_-$ that $\mathbf{w}_- \in \ker(A - \lambda_- I)^2$ but $\mathbf{w}_- \notin \ker(A - \lambda_- I)$. It thus follows that $(A - \lambda_-)\mathbf{w}_- = c\mathbf{v}_- \in \ker(A - \lambda_- I)$ for some nonzero $c \in \mathbb{C}$. By choosing $\mathbf{w}_-/c$ instead of $\mathbf{w}_-$ from the beginning (which would also have been fine) we would have obtained $c = 1$. So without loss of generality we may choose $\mathbf{w}_-$ such that

$$(A - \lambda_- I)\mathbf{w}_- = \mathbf{v}_- \quad \Leftrightarrow \quad A\mathbf{w}_- = \lambda_- \mathbf{w}_- + \mathbf{v}_-.$$

   Since A is a real matrix it then follows (by taking conjugate of the above result) that

$$A\mathbf{w}_+ = \lambda_+ \mathbf{w}_+ + \mathbf{v}_+.$$

   Finally we need to go back to $\mathbb{R}^4$, which means that we need to find out which real vectors in $\mathbb{R}^4$ can be constructed from our basis vectors $\mathbf{v}_\pm$ and $\mathbf{w}_\pm$. We have seen how to do this in the course notes before. Simplest way seems to be to use the 4 real vectors $e_- = (\mathbf{v}_- + \overline{\mathbf{v}}_-)$, $e_+ = i(\mathbf{v}_- - \overline{\mathbf{v}}_-)$, $f_- = (\mathbf{w}_- + \overline{\mathbf{w}}_-)$ and $f_- = i(\mathbf{w}_- - \overline{\mathbf{w}}_-)$.

   Then by writing out and using $\lambda_\pm = \alpha \pm i\beta$ we obtain $Ae_\pm = \alpha e_\pm \mp \beta e_\mp$, $Af_\pm = \alpha f_\pm - \beta f_\mp + e_\pm$. Writing the basis vectors as

$$e_- = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad e_+ = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad f_- = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad f_+ = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

   we obtain the desired matrix expression.

   (b) 1. One way to obtain this is by using the complex Jordan normal form that we already (possibly unknowingly) derived in part (a), where it is not so difficult to explicitly calculate the exponential from the infinite sum. Using the basis

$$\mathbf{v}_- = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{v}_+ = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{w}_- = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{w}_+ = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

   from the formulas in (a) we obtain the following matrix representation, which we will call B:

$$B = \begin{pmatrix} \lambda_- & 1 & 0 & 0 \\ 0 & \lambda_- & 0 & 0 \\ 0 & 0 & \lambda_+ & 1 \\ 0 & 0 & 0 & \lambda_+ \end{pmatrix}.$$

From this blockdiagonal form, it is not difficult to see that $\exp(At)$ is again blockdiagonal with the blocks $\left(\begin{smallmatrix} \lambda & 1 \\ 0 & \lambda \end{smallmatrix}\right)$ replaced by $\left(\begin{smallmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{smallmatrix}\right)$, thus

$$\exp(Bt) = \begin{pmatrix} e^{\lambda_- t} & te^{\lambda_- t} & 0 & 0 \\ 0 & e^{\lambda_- t} & 0 & 0 \\ 0 & 0 & e^{\lambda_+ t} & te^{\lambda_+ t} \\ 0 & 0 & 0 & e^{\lambda_+ t} \end{pmatrix}.$$

Using the result in Question 1(c) of Exercise sheet 1, we find that we can use the coordinate transformation to the real basis $\{e_-, e+, f_-, f_+\}$ of (a), to obtain $\exp(At)$ as $\exp(At) = P^{-1}\exp(Bt)P$ where

$$P = \begin{pmatrix} 1 & i & 0 & 0 \\ 0 & 0 & 1 & i \\ 1 & -i & 0 & 0 \\ 0 & 0 & 1 & -i \end{pmatrix}$$

yielding

$$\exp(At) = \exp(\alpha t)\begin{pmatrix} \cos(\beta t) & \sin(\beta t) & t\cos(\beta t) & t\sin(\beta t) \\ -\sin(\beta t) & \cos(\beta t) & -t\sin(\beta t) & t\cos(\beta t) \\ 0 & 0 & \cos(\beta t) & \sin(\beta t) \\ 0 & 0 & -\sin(\beta t) & \cos(\beta t) \end{pmatrix}$$

2. An alternative way to compute $\exp(At)$ is to realize that A has a block structure that can be exploited. We can write

$$A = \begin{pmatrix} R & I_2 \\ 0_2 & R \end{pmatrix},$$

where $R = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$, $0_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then, since these 2×2 matrices commute with each other, the computation of iterates of A is very similar as the computation of iterates of the matrix $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, so that

$$A^k = \begin{pmatrix} R^k & kR^{k-1} \\ 0 & R^k \end{pmatrix} \quad \text{and} \quad \exp(At) = \begin{pmatrix} \exp(Rt) & t\exp(Rt) \\ 0 & \exp(Rt) \end{pmatrix}.$$

The result follows directly using the form of $\exp(Rt)$ already computed in the lecture.

(c) If $\alpha < 0$ we have $\exp(At) \to 0$ (zero matrix) since for all $n > 0$, $\lim_{t\to\infty} \exp(\alpha t)t^n = 0$. Hence for all $\mathbf{x} \in \mathbb{R}^4$ we have $\lim_{t\to\infty} \exp(At)\mathbf{x} = 0$ and 0 is an asymptotically stable equilibrium point (actually even a "global" attractor).

3. A straightforward way to calculate $\exp(At)$ is to realize that it is the solution of the ODE $\dot{\mathbf{x}} = A\mathbf{x}$. We write

$$\dot{x}_4 = -3x_4, \quad \dot{x}_3 = -3x_3 + x_4, \quad \dot{x}_2 = -3x_2 + x_3, \quad \dot{x}_1 = -3x_1 + x_2.$$

We solve them one-by-one:

$$\begin{aligned}
&x_4(t) = e^{-3t}x_4(0), \\
&\dot{x}_3(t) = -3x_3 + e^{-3t}x_4(0) \quad \Leftrightarrow \quad x_3(t) = e^{-3t}x_3(0) + te^{-3t}x_4(0), \\
&\dot{x}_2(t) = -3x_2 + e^{-3t}x_3(0) + te^{-3t}x_4(0) \quad \Leftrightarrow \quad x_2(t) = e^{-3t}x_2(0) + te^{-3t}x_3(0) + \frac{t^2}{2}e^{-3t}x_4(0), \\
&\dot{x}_1(t) = -3x_1 + e^{-3t}x_2(0) + te^{-3t}x_3(0) + \frac{t^2}{2}e^{-3t}x_4(0) \quad \Leftrightarrow \\
&\qquad x_1(t) = e^{-3t}x_1(0) + te^{-3t}x_2(0) + \frac{t^2}{2}e^{-3t}x_3(0) + \frac{t^3}{6}e^{-3t}x_4(0).
\end{aligned}$$

From this, we construct the flow matrix as

$$\exp(At) = \begin{pmatrix} e^{-3t} & te^{-3t} & \frac{1}{2}t^2e^{-3t} & \frac{1}{6}t^3e^{-3t} \\ 0 & e^{-3t} & te^{-3t} & \frac{1}{2}t^2e^{-3t} \\ 0 & 0 & e^{-3t} & te^{-3t} \\ 0 & 0 & 0 & e^{-3t} \end{pmatrix}.$$

In order to understand the behaviour of $\frac{|\exp(At)\mathbf{x}|}{|\mathbf{x}|}$ when $t \geq 0$, we note show that for any $\varepsilon > 0$ there exists a $c \in \mathbb{R}$ such that $t^n e^{-3t} < ce^{-(3-\varepsilon)t}$. Namely, $t^n e^{-3t} - ce^{-(3-\varepsilon)t} = e^{-3t}(t^n - ce^{\varepsilon t})$. As the exponential grows faster as t^n, there is a $t_0(c)$ such that $(t^n - ce^{\varepsilon t}) < 0$ for all $t > t_0$. By choosing c large enough we obtain $t_0(c) \leq 0$ so that the inequality holds for all $t \geq 0$. We quantify the latter assertion: the desired inequality implies

$$n \ln(t) < \ln(c) + \varepsilon t \;\Leftrightarrow\; n \ln(t) - \varepsilon t < \ln(c).$$

The left hand side of this equation is a function of t that has a maximum at $t = n/\varepsilon$ (set derivative wrt t to zero and verify that the 2nd derivative is negative at this point). We must choose c such that $\ln(c)$ is larger than the maximum of the left hand side:

$$n \ln(\frac{n}{\varepsilon}) - n < \ln(c) \;\Leftrightarrow\; n \ln(\frac{n}{e\varepsilon}) < \ln(c) \;\Leftrightarrow\; \left(\frac{n}{e\varepsilon}\right)^n < c.$$

This thus provides a concrete estimate on the minimal size of c.

This implies that we can bound all matrix coefficients with $Ce^{-(3-\varepsilon)t}$ for some $C \in \mathbb{R}$ which in turn implies (see before) that the matrix norm is bounded by some $De^{-(3-\varepsilon)t}$ for some $D \in \mathbb{R}$. As the matrix norm is defined by the supremum of $\frac{|\exp(At)\mathbf{x}|}{|\mathbf{x}|}$ we have proven the desired bound, where exponent $\mu = 3 - \varepsilon$ and $D(\varepsilon)$ where $\varepsilon > 0$ can be as small as one likes (but of course $\lim_{\varepsilon \to 0} D(\varepsilon) \to \infty$).

4. We have $\mathbf{w}_1, \ldots, \mathbf{w}_k$ as the basis of the orthogonal complement of $\ker P$ and $\mathbf{u}_1, \ldots, \mathbf{u}_k$ as the basis of the range of P. We define the matrices $B = (\mathbf{w}_1, \ldots, \mathbf{w}_k)$ and $A = \mathbf{u}_1, \ldots \mathbf{u}_m$.

   We note that since $P^2 = P$, it follows that P acts as the identity map on its range, and hence we have $\mathbb{R}^m = \ker P \oplus \mathrm{Im}P$.

   In order for the formula for the projection to make sense, we first verify that the $k \times k$ matrix $B^\top A$ is invertible. We write

$$B^\top A = \begin{pmatrix} \mathbf{w}_1^\top \\ \vdots \\ \mathbf{w}_k^\top \end{pmatrix} (\mathbf{u}_1, \ldots, \mathbf{u}_k)$$

   Suppose that $\ker B^\top A$ contains a vector $\mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_k \end{pmatrix} \in \mathbb{R}^k \setminus \{0\}$, then

$$B^\top A\mathbf{v} = \begin{pmatrix} \mathbf{w}_1^\top \\ \vdots \\ \mathbf{w}_k^\top \end{pmatrix} \left(\sum_{i=1}^k v_i \mathbf{u}_i \right) = 0.$$

   The latter implies that there exists a nonzero vector in $\mathrm{Im}P$ such that it is orthogonal to the orthogonal complement of $\ker P$, in other words inside $\ker P$. But we have already noted that $\ker P$ and $\mathrm{Im}P$are complementary so that they do not share a common nonzero vector.

   We need to verify the properties of the projection $P = A(B^\top A)^{-1}B^\top$:

   - $P^2 = A(B^\top A)^{-1}B^\top A(B^\top A)^{-1}B^\top = A(B^\top A)^{-1}B^\top = P$.
   - We first remark that by the definition of P we have $\ker B^\top \subseteq \ker P$. Since $B^\top \mathbf{v} = (\mathbf{w}_1^\top \mathbf{v}, \ldots, \mathbf{w}_m^\top \mathbf{v})^\top$, by the definition of the vectors $\mathbf{w}_i$ we also find that if $\mathbf{v} \in \ker P$ then also $\mathbf{v} \in \ker B^\top$, i.e. $\ker P \subseteq \ker B^\top$. Hence $\ker B^\top = \ker P$, as desired.

- $B^\top P\mathbf{v} = B^\top A(B^\top A)^{-1}B^\top \mathbf{u}_i = B^\top \mathbf{v}$ for any $\mathbf{v} \in \mathbb{R}^m$. In order to show that indeed $P\mathbf{u} = \mathbf{u}$ for all $\mathbf{u} \in \mathrm{Im}P$ we argue that $B^\top$ is invertible on $\mathrm{Im}P$. If this was not the case, $\ker B^\top$ would have a nontrivial vector common with $\mathrm{Im}P$, which is not the case since $\ker B^\top = \ker P$.

In the case of a projection on the complex vector space $\mathbb{C}^m$, we need to replace the *transpose* of B by the transpose of the complex conjugate: $P = A(\bar{B}^\top A)^{-1}\bar{B}^\top$.

The proof is similar, but we need to recall that the inner product (defining when a vector is perpendicular to another) involves a complex conjugate $\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^m x_i \bar{y}_i$.

5. (a) If $M = N_n N_{n'}$, the matrix coefficient $M_{i,k}$ is equal to

$$M_{i,k} = \sum_{j=1}^m (N_n)_{i,j}(N_{n'})_{j,k} = (N_n)_{i,i+n}(N_{n'})_{i+n,i+n+n'} = \begin{cases} 0 & \text{if } k \neq i+n+n' \\ & \text{or } k = i+n+n' \geq m, \\ 1 & \text{if } k = i+n+n' < m. \end{cases}$$

so that indeed $M_{i,k} = (N_{n+n'})_{i,k}$. If $n + n' \geq m$ then consequently $M = N_{n+n'} = 0$.

(b) From (a) it follows that $N_j^k = N_{k\cdot j}$ which is equal to the zero matrix if $k \cdot j \geq m$. Hence

$$\exp(N_j) = \sum_{k=0}^\infty \frac{N_j^k}{k!} = \sum_{k=\mathbb{N}, k\cdot j<m} \frac{N_j^k}{k!}.$$

(c) As I and N_1 commute, we have

$$\exp(Mt) = \exp(\lambda t)\exp(tN_1) = \exp(\lambda t)\sum_{k=0}^{m-1}\frac{t^k}{k!}N_k = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} & \cdots & \frac{t^{m-1}}{(m-1)!}e^{\lambda t} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & te^{\lambda t} \\ 0 & \cdots & 0 & e^{\lambda t} \end{pmatrix}.$$

(d) The solution for Question 3 (with real eigenvalue 3) follows immediately from the above formula. In the case of complex eigenvalues $\lambda = \alpha \pm i\beta$, defining

$$R_\lambda = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}, \quad \text{and} \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

the Jordan normal form admits the decomposition

$$M = \begin{pmatrix} R_\lambda & I & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & I \\ 0 & \cdots & \cdots & 0 & R_\lambda \end{pmatrix} = \begin{pmatrix} R_\lambda & 0 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & R_\lambda \end{pmatrix} + \begin{pmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & I \\ 0 & \cdots & \cdots & 0 & 0 \end{pmatrix}$$

where the matrices on the right hand side commute. Hence the exponential is the product of the exponentials of the matrices on the right hand side. We have $\exp(R_\lambda t) = e^{\alpha t}\hat{R}_\beta$ with $\hat{R}_\beta = \begin{pmatrix} \cos(\beta t) & \sin(\beta t) \\ -\sin(\beta t) & \cos(\beta t) \end{pmatrix}$ and

$$\exp(t \begin{pmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & I \\ 0 & \cdots & \cdots & 0 & 0 \end{pmatrix}) = \begin{pmatrix} I & tI & \cdots & \frac{t^{m-1}}{(m-1)!}I \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & tI \\ 0 & \cdots & \cdots & I \end{pmatrix}$$

so that

$$\exp(Mt) = e^{\alpha t} \begin{pmatrix} \hat{R}_\beta & t\hat{R}_\beta & \cdots & \frac{t^{m-1}}{(m-1)!}\hat{R}_\beta \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & t\hat{R}_\beta \\ 0 & \cdots & \cdots & \hat{R}_\beta \end{pmatrix}$$

One readily verifies that the answer of Question 1 on Exercise sheet 2 has this form.

6. Let $T \in Gl(m, \mathbb{R})$, then TST^{-1} is semi-simple if and only if S is, and TNT^{-1} is nilpotent if and only if N is. Hence, without loss of generality we may consider the coordinate frame for which a matrix $M \in gl(m\mathbb{R})$ is in Jordan normal form. As in Question 2 above, every Jordan block can be written as the sum of a semi-simple matrix S which is (block-)diagonal and an off-diagonal nilpotent matrix N. It is readily verified that these matrices commute, i.e. $SN = NS$. As $\mathbb{R}^m$ can be decomposed into generalized eigenspaces E_λ that are M-invariant (and hence also S- and N-invariant), without loss of generality we focus on the case that the domain is equal to one generalized eigenspace (so in what follows we let $\lambda \in \mathbb{R}$ but in the case of complex eigenvalues the result follows in a similar way by viewing the matrix temporarily as a complex one [where in fact the Jordan-Chevalley decomposition also holds]).

   We have at least one decomposition into semi-simple and nilpotent part $M = S + N$ equal to the Jordan normal form. To show it is unique, suppose that $M = \hat{S} + \hat{N}$ is another such decomposition.

   Since M commutes with both S and $\hat{S}$, we find that M, S and $\hat{S}$ have the same generalized eigenspaces. Since S is (complex) diagonalizable, we observe that S and $\hat{S}$ are simultaneously (complex) diagonalizable from which it follows that S and $\hat{S}$ commute and that $S - \hat{S}$ is also (complex) diagonalizable and thus also semi-simple. From the fact that S and $\hat{S}$ are simultaneously diagonalizable we also deduce that N and $\hat{N}$ commute: namely, $(N + S)(\hat{N} + \hat{S}) = (\hat{N} + \hat{S})(N + S) \Leftrightarrow N\hat{N} = \hat{N}N$ in the coordinate frame in which S and $\hat{S}$ are simultaneously diagonal, and hence this relation holds in any coordinate frame. We now consider $N - \hat{N}$, and observe that this is nilpotent since if $N^p = \hat{N}^p = 0$ then $(N - \hat{N})^{2p} = 0$. We now use the fact that $M = S + N = \hat{S} + \hat{N}$ to derive that $S - \hat{S} = N - \hat{N}$. Since only the 0 matrix has the property that it is both semi-simple and nilpotent at the same time, we obtain that $S = \hat{S}$ and $N = \hat{N}$.

   Let $M = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ then $M = S + N$ with

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

   where S is semi-simple and N is nilpotent. Of course, M is also nilpotent, so the decomposition of M into nilpotent and semi-simple parts here is not unique (if we do not insist on these to commute with each other). One easily verifies that with the above choices $SN \neq NS$.

7. (a) (i) A_1 has eigenvalues $\pm 2i$ so A_1 is semi-simple and J-C decomposition is $A_1 = A_1 + 0$. Real Jordan normal form is $\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$. A_2 has double eigenvalue 8 but only one eigenvector $(1, 2)^T$. So A_2 is not semi-simple and the real Jordan normal form is $\begin{pmatrix} 8 & 1 \\ 0 & 8 \end{pmatrix}$. Since in this case the semi-simple part must be a multiple of the identity matrix, the J-C decomposition is given by $A_2 = S + N$ where $S = 8\mathrm{I}$ and $N = \begin{pmatrix} -2 & 1 \\ -4 & 2 \end{pmatrix}$. Clearly $SN = NS$ and also $N^2 = 0$.

   (ii) The complex eigenvectors for the eigenvalues $\pm 2i$ are $(1, \frac{4\pm 2i}{5})^T$. Thus

$$\exp(At) = P \begin{pmatrix} e^{2it} & 0 \\ 0 & e^{-2it} \end{pmatrix} P^{-1}$$

   where

$$P = \begin{pmatrix} 1 & 1 \\ \frac{4+2i}{5} & \frac{4-2i}{5} \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} \frac{1}{2} + i & -\frac{5}{4}i \\ \frac{1}{2} - i & \frac{5}{4}i \end{pmatrix}$$

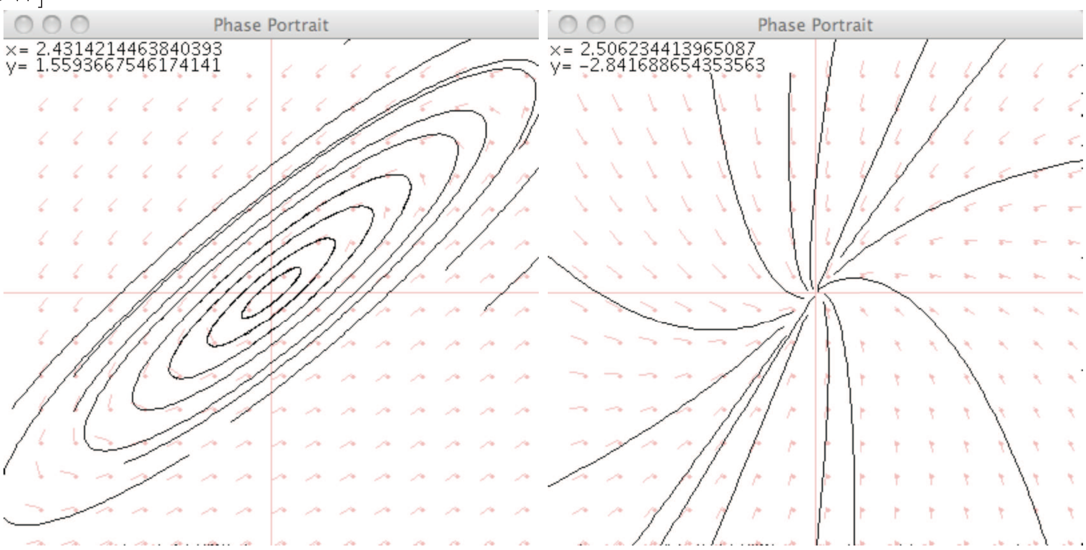
yielding

$$\exp(A_1 t) = \begin{pmatrix} \cos(2t) - 2\sin(2t) & (5/2)\sin(2t) \\ -2\sin(2t) & \cos(2t) + 2\sin(2t) \end{pmatrix}.$$

Using the J-C decomposition of A_2 we find $\exp(St) = e^{8t}\mathrm{I}$ and $\exp(Nt) = \mathrm{I} + Nt$ so that

$$\exp(A_2 t) = e^{8t}(\mathrm{I} + Nt) = e^{8t} \begin{pmatrix} 1 - 2t & t \\ -4t & 1 + 2t \end{pmatrix}$$

Sketches of the phase portraits (anything roughly close should do): [PLEASE NOTE THAT ARROWS SHOULD BE ADDED TO SKETCHED SOLUTION CURVES INDICATE DIRECTION OF THE FLOW]



(b) (i) $\ker P_1$ is the eigenspace of A for eigenvalue 4.

(ii) P_1 is the projection to the eigenspace of A for eigenvalue -1 along the direction of the eigenspace for eigenvalue 4. P_2 is the projection to the eigenspace of A for eigenvalue 4 along the direction of the eigenspace for eigenvalue -1. Hence $\mathrm{range} P_2 = \ker P_1$ and $\mathrm{range} P_1 = \ker P_2$.

(iii) $A = \frac{d}{dt}\exp(At)|_{t=0} = -P_1 + 4P_2 = \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}$.