M2AA1 Differential Equations
Exercise sheet 2

1. Consider a nonautonomous ODE $\dot{\mathbf{x}} = f(\mathbf{x}, t)$ with $\mathbf{x} \in \mathbb{R}^m$.

   (a) Show that by extending the phase space from $\mathbb{R}^m$ to $\mathbb{R}^m \times \mathbb{R}$ this ODE can be rewritten as an autonomous ODE.

   (b) Derive a relationship between the vector field f and the flow map $\Phi^{t_1,t_0} : \mathbb{R}^m \to \mathbb{R}^m$ associated to the nonautonomous ODE, so that $\Phi^{t_1,t_0}\mathbf{x}(t_0) = \mathbf{x}(t_1)$ for any solution $\mathbf{x}(t)$ of this ODE.

   (c) Calculate the flow of the nonautonomous ODE

$$\begin{cases} \frac{d}{dt}x &= 2x + y - t \\ \frac{d}{dt}y &= 2x - y + t \end{cases}$$

2. Consider a linear autonomous vector field in $\mathbb{R}^4$, $\frac{d\mathbf{x}}{dt} = A\mathbf{x}$, whose eigenvalues $\lambda = \alpha \pm i\beta$ are complex with algebraic multiplicity two and geometric multiplicity one.

   (a) Show that one can choose coordinates so that A takes the (Jordan normal) form

$$A = \begin{pmatrix} \alpha & \beta & 1 & 0 \\ -\beta & \alpha & 0 & 1 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -\beta & \alpha \end{pmatrix}$$

   (b) Compute the flow $\Phi^t = \exp(At)$ (with the coordinate choice of (a)). Hint: use the complex Jordan normal form of A.

   (c) Show that the equilibrium 0 is asymptotically stable if $\alpha < 0$.

3. With

$$A = \begin{pmatrix} -3 & 1 & 0 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & -3 \end{pmatrix},$$

   calculate the flow $\exp(At)$ by explicitly integrating the flow $\dot{\mathbf{x}} = A\mathbf{x}$. Show that there exist positive constants c and μ, such that

$$|\exp(At)\mathbf{x}| < c\exp(-\mu t)|\mathbf{x}|, \quad \forall t \geq 0.$$

4. Prove the following proposition

   Proposition 1 (Projection). *A formula for the $m \times m$ matrix representing the projection in $\mathbb{R}^m$ with a given range and kernel can be found as follows. Let the vectors $\mathbf{u}_1, \dots, \mathbf{u}_k$ form a basis for the range of the projection, and assemble these vectors in the $m \times k$ matrix A. The range and the kernel are complementary spaces, so the kernel has dimension $m - k$. It follows that the orthogonal complement of the kernel has dimension k. Let $\mathbf{w}_1, \dots, \mathbf{w}_k$ form a basis for the orthogonal complement of the kernel of the projection, and assemble these vectors in the matrix B. Then the projection to the range is given by*

$$P = A(B^\top A)^{-1}B^\top.$$

   Show that the projection takes the form $P = A(\bar{B}^\top A)^{-1}\bar{B}^\top$ if the vector space is complex ($\mathbb{C}^m$)? (Here $\bar{B}$ denotes the complex conjugate of B.)

5. Let $N_n \in gl(m, \mathbb{R})$, $n \in \{0, 1, \dots, k-1\}$ be the $m \times m$ matrix with coefficients $N_{i,j}$ where $N_{i,i+n} = 1$ for $i = 1, \dots, m-n$ and $0 \leq n \leq m-1$ and $N_{i,j} = 0$ whenever $j - i \neq n$. For instance, with $m = 3$ we have $N_1 = \left(\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{smallmatrix}\right)$. (Hint: To develop some intuition, although the following questions are formulated for general m, first answer them in the case that $m = 3$ and/or $m = 4$.)

(a) Show that $N_j N_k = N_{j+k}$ if $j + k < m$ and $N_j N_k = 0$ if $j + k \geq m$.

(b) Compute $\exp(N_j)$.

(c) Consider a matrix $M \in gl(m, \mathbb{R})$ which is a Jordan block with real eigenvalue λ. Thus $M = \lambda I + N_1$. Use this observation to compute $\exp(Mt)$ with $t \in \mathbb{R}$.

(d) Use the result in (c) to compute $\exp(At)$, for the cases that the matrix A is as in Question 2 and Question 3 above.

6. A matrix is called *semi-simple* if the geometric multiplicity of each of its eigenvalues equals its algebraic multiplicity. Semi-simple matrices have a (block-)diagonal Jordan normal form. A matrix N is called *nilpotent* if there exists a $k \in \mathbb{N}$ such that its kth iterate equals the zero matrix, i.e. $N^k = 0$. Prove the following theorem (that underlies the calculations in Question 5 above):

Theorem 2 (Jordan-Chevalley decomposition)**.** *Any $M \in gl(m, \mathbb{R})$ has a unique decomposition $M = S + N$ where S is semi-simple, N is nilpotent and $NS = SN$.*

(Hint: use the fact that in Jordan normal form this decomposition is easy to find, cf. Question 5.)

Can you give an example of the decomposition of a 2×2 matrix M into a semi-simple S and nilpotent N such that S and N do not commute? This to illustrate that the condition that $SN = NS$ is essential for the "uniqueness" of the decomposition.

7. (2008 exam)

(a) Let

$$A_1 = \begin{pmatrix} -4 & 5 \\ -4 & 4 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 6 & 1 \\ -4 & 10 \end{pmatrix}.$$

(i) Give the <u>real Jordan normal form</u> and <u>Jordan-Chevalley decomposition</u> of A_i, for $i = 1, 2$. [Note: You need <u>not</u> derive the coordinate transformation that transforms the matrices into Jordan normal form.]

(ii) Calculate the flow $\Phi^t : \mathbb{R}^2 \to \mathbb{R}^2$ of the ODE

$$\frac{d\mathbf{x}}{dt} = A_i \mathbf{x}, \text{ for } i = 1, 2$$

and sketch the phase portraits. [Hint: In the calculation of the flow, for A_1 it may be convenient to use the fact that the matrix is conjugate to its Jordan normal form, and for A_2 it may be convenient to use the Jordan-Chevalley decomposition.]

- (b)] Let $A \in gl(2, \mathbb{R})$ be such that

$$\exp(At) = e^{-t} P_1 + e^{4t} P_2,$$

where

$$P_1 = \begin{pmatrix} \frac{3}{5} & -\frac{3}{5} \\ -\frac{2}{5} & \frac{2}{5} \end{pmatrix} \text{ and } P_2 = \begin{pmatrix} \frac{2}{5} & \frac{3}{5} \\ \frac{2}{5} & \frac{3}{5} \end{pmatrix}.$$

(i) Discuss the relationship between $\ker P_1$ and A.

(ii) Explain why $P_1 P_2 = P_2 P_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

(iii) Calculate A.

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