M2AA1 Differential Equations
Exercise sheet 1 answers

1. Flow: $\Phi^t : \mathbb{R} \to \mathbb{R}$, $\Phi^t(x) = e^{-0.5t}x$. $|\Phi^t([a,b])| = |e^{-0.5t}[a,b]| < |[a,b]|$ if $t > 0$.

2. General answer $\Phi^t(\mathbf{x}) = \exp(tL)\mathbf{x}$. If you work this out (either by using Taylor expansion of exp, or by solving ODE by different method) , you find that $\exp(tL)$ is equal to
(a) $\begin{pmatrix} e^t & \sinh(t) \\ 0 & e^{-t} \end{pmatrix}$, (b) $\begin{pmatrix} e^t & te^t \\ 0 & e^t \end{pmatrix}$, (c) $\begin{pmatrix} \cos(at) & \sin(at) \\ -\sin(at) & \cos(at) \end{pmatrix}$.

3. (a) We note that $A = PJP^{-1}$ where
$$J = \begin{pmatrix} -6 & 0 \\ 0 & 4 \end{pmatrix}, \quad P = \frac{1}{2}\begin{pmatrix} 3 & -1 \\ -1 & 1 \end{pmatrix}.$$
Note that the columns of P represent the eigenvectors of A. Thus
$$\exp(A) = P\exp(J)P^{-1} = P\begin{pmatrix} e^{-6} & 0 \\ 0 & e^4 \end{pmatrix}P^{-1} = \frac{1}{2}\begin{pmatrix} 3e^{-6} - e^4 & -3e^4 + 3e^{-6} \\ e^4 - e^{-6} & -e^{-6} + 3e^4 \end{pmatrix}$$
(b) We note that $A = PJP^{-1}$ where
$$J = \begin{pmatrix} -6 & 0 \\ 0 & 4 \end{pmatrix}, \quad P = \frac{1}{8}\begin{pmatrix} 1 & -1 \\ 0 & 8 \end{pmatrix}.$$
Thus
$$\exp(A) = P\exp(J)P^{-1} = P\begin{pmatrix} e^{-7} & 0 \\ 0 & e \end{pmatrix}P^{-1} = \begin{pmatrix} e^{-7} & \frac{-e+e^{-7}}{8} \\ 0 & e \end{pmatrix}.$$

4. The eigenvalues of L are $\lambda_1 = -1, \lambda_2 = 2, \lambda_3 = 3$ and the corresponding eigenvectors are (respectively)
$$\mathbf{v}_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}.$$
Let P_1 denote the projection to the line $\langle \mathbf{v}_1 \rangle$ along the plane $\langle \mathbf{v}_2, \mathbf{v}_3 \rangle$ with analogous definitions of projections P_2 and P_3 to $\langle \mathbf{v}_2 \rangle$ and $\langle \mathbf{v}_3 \rangle$, so that any $\mathbf{x} \in \mathbb{R}^3$ can be written (uniquely) as $\mathbf{x} = P_1\mathbf{x} + P_2\mathbf{x} + P_3\mathbf{x}$. Let $\mathbf{x}_i := P_i\mathbf{x}$, then the corresponding solution of the ODE is
$$\mathbf{x}(t) = \sum_{i=1}^{3} e^{\lambda_i t} P_i \mathbf{x}.$$
To work out the projections P_i we use the general expression given in the course notes (see also problem sheet 2, question 4). We derive in detail the projection P_1. Note that P_1 is defined by $P_1^2 = P_1$ and specification of kernel and range. P_1 must be so that $\ker P_1 = \langle \mathbf{v}_2, \mathbf{v}_3 \rangle$ and of course range$P_1 = \langle \mathbf{v}_1 \rangle$. Let A be defined a matrix whose columns form a basis of the range of P_1, so here $A = \mathbf{v}_1$. Let B be defined as a matrix whose columns form a basis of the orthogonal complement of the kernel of the projection. The latter basis can for instance be chosen as
$$(\ker P_1)^\perp = \langle \mathbf{v}_2, \mathbf{v}_3 \rangle^\perp = \langle \begin{pmatrix} -3 \\ 1 \\ 9 \end{pmatrix} \rangle,$$
so B may be chosen as $\begin{pmatrix} -3 \\ 1 \\ 9 \end{pmatrix}$. Then, as $B^T A = 12$, we obtain
$$P_1 = A(B^T A)^{-1} B^T = \frac{1}{12}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\begin{pmatrix} -3 & 1 & 9 \end{pmatrix} = \frac{1}{12}\begin{pmatrix} 3 & -1 & -9 \\ 0 & 0 & 0 \\ -3 & 1 & 9 \end{pmatrix}.$$

Expressions for P_2 and P_3 can be obtained in a similar way, and we obtain

$$\begin{aligned}\exp(tL) &= e^{-t}P_1 + e^{2t}P_2 + e^{3t}P_3 \\ &= e^{-t}\begin{pmatrix} \frac{1}{4} & -\frac{1}{12} & -\frac{3}{4} \\ 0 & 0 & 0 \\ -\frac{1}{4} & \frac{1}{12} & \frac{3}{4} \end{pmatrix} + e^{2t}\begin{pmatrix} 0 & -\frac{2}{3} & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{3} & 0 \end{pmatrix} + e^{3t}\begin{pmatrix} \frac{3}{4} & \frac{3}{4} & \frac{3}{4} \\ 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix} \\ &= \begin{pmatrix} \frac{3}{4}e^{3t} + \frac{1}{4}e^{-t} & \frac{3}{4}e^{3t} - \frac{2}{3}e^{2t} - \frac{1}{12}e^{-t} & \frac{3}{4}e^{3t} - \frac{3}{4}e^{-t} \\ 0 & e^{2t} & 0 \\ -\frac{1}{4}e^{-t} + \frac{1}{4}e^{3t} & \frac{1}{4}e^{3t} - \frac{1}{3}e^{2t} + \frac{1}{12}e^{-t} & \frac{3}{4}e^{-t} + \frac{1}{4}e^{3t} \end{pmatrix}\end{aligned}$$

Solution curves through the phase space points are obtained by direct application of the flow. Please note the relationship to the decomposition of the initial condition vector into eigenvectors of L.

$$\begin{aligned}
&\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \tfrac{7}{12}\mathbf{v}_1 - \tfrac{1}{3}\mathbf{v}_2 + \tfrac{3}{4}\mathbf{v}_3 \quad \text{and} \quad \exp(Lt)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -(7/12)e^{-t} + (9/4)e^{3t} - (2/3)e^{2t} \\ e^{2t} \\ (3/4)e^{3t} + (7/12)e^{-t} - (1/3)e^{2t}) \end{pmatrix} \\
&\begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix} = 2\mathbf{v}_1 \quad \text{and} \quad \exp(Lt)\begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} -2e^{-t} \\ 0 \\ 2e^{-t} \end{pmatrix} \\
&\begin{pmatrix} 5 \\ -3 \\ 2 \end{pmatrix} = \mathbf{v}_2 + \mathbf{v}_3 \quad \text{and} \quad \exp(Lt)\begin{pmatrix} 5 \\ -3 \\ 2 \end{pmatrix} = \begin{pmatrix} 3e^{3t} + 2e^{2t} \\ -3e^{2t} \\ e^{3t} + e^{2t} \end{pmatrix}.
\end{aligned}$$

The first is neither on E^u nor on E^s, the second is on E^s and the third on E^u.

5. (a) $E^u = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle$, $E^s = \left\langle \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\rangle$, $E^c = 0$. (b) $E^u = \mathbb{R}^2$, $E^c = 0$, $E^s = 0$. (c) $E^c = \mathbb{R}^2$, $E^u = 0$, $E^s = 0$.

6. (a) Use binomial formula and commutation A with B to obtain

$$\exp(A+B) = \sum_{k=0}^{\infty} \frac{(A+B)^k}{k!} = \sum_{k=0}^{\infty}\sum_{\ell=0}^{k} \frac{A^{k-\ell}B^{\ell}}{(k-\ell)!\ell!} = \left(\sum_{k=0}^{\infty}\frac{A^k}{k!}\right)\cdot\left(\sum_{\ell=0}^{\infty}\frac{B^\ell}{\ell!}\right).$$

   (b) follows directly from (a), with $B = -A$.

   (c) Follows from the fact that $(BAB^{-1})^k = BA^kB^{-1}$.

   (d) If $Av = \lambda v$ then $A^k v = \lambda^k v$ and $\exp(A)v = \sum_{k=0}^{\infty}\frac{A^k v}{k!} = \sum_{k=0}^{\infty}\frac{\lambda^k}{k!}v = \exp(\lambda)v$.

7. For example $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ we have $\exp(A) = \begin{pmatrix} e^1 & 0 \\ 0 & e^{-1} \end{pmatrix}$ and $\exp(B) = \begin{pmatrix} \cos(1) & -\sin(1) \\ \sin(1) & \cos(1) \end{pmatrix}$ so that

$$\exp(A)\exp(B) = \begin{pmatrix} e^1\cos(1) & -e^1\sin(1) \\ e^{-1}\sin(1) & e^{-1}\cos(1) \end{pmatrix} \neq \exp(B)\exp(A) = \begin{pmatrix} e^1\cos(1) & -e^{-1}\sin(1) \\ e^1\sin(1) & e^{-1}\cos(1) \end{pmatrix}.$$

8. The solutions of the ODE $d\mathbf{x}/dt = A\mathbf{x}$ with "initial" condition $\mathbf{x}(0) = \mathbf{y}$ take the form $\mathbf{x}(t) = \exp(tA)\mathbf{y}$. To obtain all possible solutions we can vary $\mathbf{y} \in \mathbb{R}^m$, moreover, since the flow is linear, if $\{e_1, \ldots, e_m\}$ is a basis of $\mathbb{R}^m$ then the solution space is a linear vector space with basis $\{\exp(tA)e_1, \ldots, \exp(tA)e_m\}$, or in other words each solution $\mathbf{x}(t)$ can be written as a linear combination of these basis vectors: $\mathbf{x}(t) = \sum_{i=1}^{m} y_i \exp(At)e_i$. Thus the space of solutions is an m-dimensional (real) linear vector space.

9. We note that the norm

$$||A|| = \sup_{\mathbf{x}\in\mathbb{R}^m\setminus\{0\}} \frac{|A\mathbf{x}|}{|\mathbf{x}|}.$$

In the case of distinct eigenvalues, it is equal to the absolute value of the eigenvalue with largest modulus (absolute value). If not, we can bound the norm in other ways. Let a be the absolute value of the matrix entry A_{ij} with largest modulus and X denote the absolute value of the largest vector entry x_i of $\mathbf{x}$, then

$$|A\mathbf{x}| = \sqrt{\sum_{i=1}^{m}(\sum_{j=1}^{m} A_{ij}x_j)^2} \leq a\sqrt{\sum_{i=1}^{m}(mX)^2} = am^{3/2}X.$$

As $|x| \geq X$ we thus obtain that

$$\frac{|A\mathbf{x}|}{|\mathbf{x}|} \leq am^{3/2} < \infty.$$

Hence, a matrix $T \in gl(m,\mathbb{R})$ (an $m \times m$ matrix with real entries) is a bounded linear map: $||T|| < \infty$. Then also $||T^k|| \leq ||T||||T^{k-1}|| \leq \ldots \leq ||T||^k$. Hence $||\exp(T)|| \leq \exp(||T||)$.

version February 8, 2009