M2AA1 Differential Equations
Exercise sheet 1

1. Calculate the flow Φ^t of the ODE $\dot{x} = -0.5x$ with $x \in \mathbb{R}$ and show the image of any subinterval of the real line under the forward flow ($t > 0$) is another <u>smaller</u> subinterval.

2. Derive the flow Φ^t for the linear ODE $\dot{\mathbf{x}} = L\mathbf{x}$ with $\mathbf{x} \in \mathbb{R}^2$ for the following choices of the matrix L :
(a) $L = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$ (b) $L = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ (c) $L = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}$ with $a \neq 0$

3. Calculate $\exp(A)$ for the following choices of the matrix A :
(a) $A = \begin{pmatrix} -11 & -15 \\ 5 & 9 \end{pmatrix}$ (b) $A = \begin{pmatrix} -7 & -1 \\ 0 & 1 \end{pmatrix}$
Hint: if A can be diagonalized, it may be useful to find the (linear) coordinate transformation that achieves this diagonalization and use the property mentioned in Question 6(b).
Suggestion: you can verify your answers with Maple.

4. Derive the flow for the linear ODE $\dot{\mathbf{x}} = L\mathbf{x}$ with $\mathbf{x} \in \mathbb{R}^3$ and
$$L = \begin{pmatrix} 2 & 1 & 3 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$
Find the solution curves through the phase space points
$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} 5 \\ -3 \\ 2 \end{pmatrix}.$$

5. For the linear ODEs in questions 2 and 4, find the stable, unstable and centre subspaces E^s, E^u and E^c.

6. Use the definition of the matrix exponential to demonstrate that the following properties hold (with $A, B \in gl(m, \mathbb{R})$):
   (a) If $AB = BA$ then $\exp(A)\exp(B) = \exp(A + B)$ and $B\exp(A) = \exp(A)B$.
   (b) $\exp(-A)\exp(A) = I$ (the identity matrix).
   (c) If B is invertible, then $\exp(BAB^{-1}) = B\exp(A)B^{-1}$.
   (d) If v is an eigenvector of A with eigenvalue λ, then $\exp(A)v = \exp(\lambda)v$.

7. With reference to Question 6(a) above: find an example of a pair of matrices A, B for which $\exp(A)\exp(B) \neq \exp(A + B)$.

8. Show that the solution space of an autonomous first order ODE in $\mathbb{R}^m$ is an m-dimensional linear vector space. Argue that the solution space of a second order ODE in $\mathbb{R}$ is a two-dimensional vector space (compare this with what you know about the solutions of a damped harmonic oscillator).

9. Let the norm of a linear map $A : \mathbb{R}^m \to \mathbb{R}^m$ be defined as
$$||A|| = \sup_{\mathbf{x} \in \mathbb{R}^m \setminus \{0\}} \frac{|A\mathbf{x}|}{|\mathbf{x}|},$$
where the norm $|\mathbf{x}|$ denotes the length of a vector $\mathbf{x} \in \mathbb{R}^m$. We say that A is *bounded* if $||A|| < \infty$.
Show that the linear map from $\mathbb{R}^m$ to $\mathbb{R}^m$ induced by multiplication by a matrix $T \in gl(m, \mathbb{R})$ (an $m \times m$ matrix with real entries) is a bounded linear map.
Prove that $\exp(T)$ is a bounded linear map if $T \in gl(m, \mathbb{R})$. (Hint: find an upper bound of the norm of T in terms of its coefficient with largest modulus, and use this to find an upper bound for the norm of $\exp(T)$.)

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