Imperial College
London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2008

This paper is also taken for the relevant examination for the Associateship.

M2AA1

Specimen Paper

Date: examdate Time: examtime

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. (a) Consider the linear ODE

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}, \tag{1}$$

with $A \in gl(m, \mathbb{R})$ (an $m \times m$ matrix with real entries) and $\mathbf{x} \in \mathbb{R}^m$.

(i) Show that the ODE (1) with initial value $\mathbf{x}(0) = \mathbf{y}$ has the unique solution

$$\mathbf{x}(t) = \exp(At)\mathbf{y}.$$

[Hint: First show from first principles that $\frac{d}{dt}\exp(At) = A\exp(At)$.]

(ii) Show that, for each value of $t \in \mathbb{R}$, the map $\exp(At) : \mathbb{R}^m \to \mathbb{R}^m$ is linear and invertible.

(b) Give the definition of a contraction on a complete metric space and prove the following theorem:

Theorem: Let X be a complete metric space, and $F : X \to X$ be a contraction. Then F has a unique fixed point.

[You need not give the definition of a Cauchy sequence.]

(c) (i) Let the map T be defined by

$$T(u(t)) = u_0 + \int_{t_0}^{t} f(u(s))ds, \tag{2}$$

where $u_0 \in \mathbb{R}^m$, $f : \mathbb{R}^m \to \mathbb{R}^m$ is Lipschitz and $u : \mathbb{R} \to \mathbb{R}^m$ is continuous.
Show that fixed points of T correspond to solutions of the ODE

$$\frac{du(t)}{dt} = f(u(t)), \tag{3}$$

with initial value $u(t_0) = u_0$.

(ii) The map T defined in (2) is a contraction on a suitable complete metric space of continuous functions $u : [-a, a] \to \mathbb{R}^m$ with $u(t_0) = u_0$ and $a \neq 0$ sufficiently small. Show how this fact may be used to prove local existence and uniqueness of solutions for the ODE (3). [Please only answer the question as stated. You must NOT prove that T is a contraction and need NOT provide any further details about a or the complete metric space involved.]

2. (a) Let

$$A_1 = \begin{pmatrix} -4 & 5 \\ -4 & 4 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 6 & 1 \\ -4 & 10 \end{pmatrix}.$$

(i) Give the <u>real Jordan normal form</u> and <u>Jordan-Chevalley decomposition</u> of A_i, for $i = 1, 2$. [Note: You need <u>not</u> derive the coordinate transformation that transforms the matrices into Jordan normal form.]

(ii) Calculate the flow $\Phi^t : \mathbb{R}^2 \to \mathbb{R}^2$ of the ODE

$$\frac{d\mathbf{x}}{dt} = A_i\mathbf{x}, \text{ for } i = 1, 2$$

and sketch the phase portraits. [Hint: In the calculation of the flow, for A_1 it may be convenient to use the fact that the matrix is conjugate to its Jordan normal form, and for A_2 it may be convenient to use the Jordan-Chevalley decomposition.]

(b) Let $A \in gl(2, \mathbb{R})$ be such that

$$\exp(At) = e^{-t}P_1 + e^{4t}P_2,$$

where

$$P_1 = \begin{pmatrix} \frac{3}{5} & -\frac{3}{5} \\ -\frac{2}{5} & \frac{2}{5} \end{pmatrix} \text{ and } P_2 = \begin{pmatrix} \frac{2}{5} & \frac{3}{5} \\ \frac{2}{5} & \frac{3}{5} \end{pmatrix}.$$

(i) Discuss the relationship between $\ker P_1$ and A.

(ii) Explain why $P_1P_2 = P_2P_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

(iii) Calculate A.

3. Consider the equations of motion for a nonlinear oscillator with friction

$$\frac{d^2x}{dt^2} = -\kappa x - x^3 - \mu \frac{dx}{dt}, \tag{4}$$

where $x \in \mathbb{R}$, μ is a non-negative parameter (friction constant) and κ is a constant that can be either positive or negative (elasticity constant).

(a) Write the equation (4) as a first order ODE on the plane $\mathbb{R}^2$.

(b) (i) Show that the *energy* of this oscillator

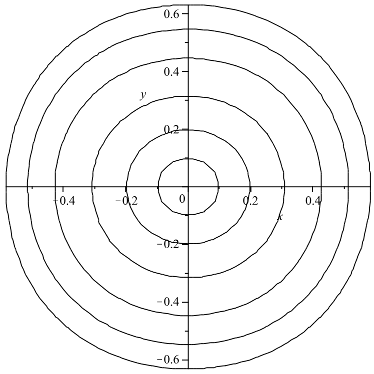
$$E = \frac{1}{2}\left(\frac{dx}{dt}\right)^2 + \frac{\kappa}{2}x^2 + \frac{1}{4}x^4$$

is a Lyapunov function of (4).

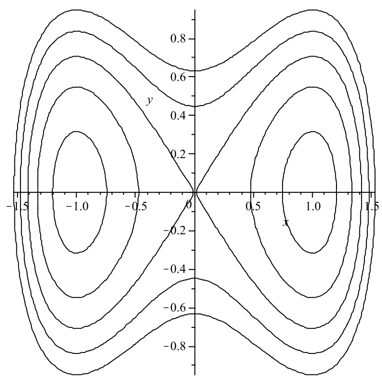
(ii) Sketch the phase portrait and describe all ω- and α-limit sets of this system in the case that:

- $\kappa = 1$ and $\mu = 0$,
- $\kappa = -1$ and $\mu > 0$.

You may use the following sketches of the contours (E =constant) of the Lyapunov function E when $\kappa = 1$ and $\kappa = -1$:



$\kappa = 1$



$\kappa = -1$

(c) (i) Analyze the bifurcation of equilibria in the system (4) as κ increases through $\kappa = 0$. Sketch the bifurcation diagram (bifurcation parameter κ versus x).

(ii) Discuss whether (and if so, how) the bifurcation diagram in (ii) changes if one adds a term εx^7 (with $|\varepsilon|$ being very small) to the right-hand-side of (4).

(d) Equilibria can be viewed as the intersection of *nullclines* of the planar vector field derived in (a). Recall that the nullclines are defined as curves on which one of the components of the vector field is equal to zero.

Show for general planar vector fields $f : \mathbb{R}^2 \to \mathbb{R}^2$, that the derivative of the vector field $Df(\mathbf{x}_0)$ at an equilibrium $\mathbf{x}_0$ is invertible if and only if the nullclines have a transverse intersection at $\mathbf{x}_0$.

Discuss this relationship in the context of conditions for persistence of equilibria under small perturbations.

4. Consider the following model of the chemical reaction between two substances whose concentrations are denoted by x and y, respectively:

$$\begin{aligned}\frac{dx}{dt} &= a - x - \frac{4xy}{1+x^2},\\ \frac{dy}{dt} &= x\left(1 - \frac{y}{1+x^2}\right).\end{aligned}$$

Here a is a <u>positive</u> parameter. Note also that as x and y represent concentrations, we are only interested in $x, y \geq 0$. The model serves to illustrate that chemical reactions may yield asymptotic solutions that oscillate instead of being stationary.

(a) (i) Show that there is a unique equilibrium and that at this equilibrium the derivative of the vector field (Jacobian) is equal to

$$\frac{1}{25+a^2}\begin{pmatrix} -125+3a^2 & -20a \\ 2a^2 & -5a \end{pmatrix}.$$

(ii) Show that the equilibrium is asymptotically stable if $a < \frac{5}{6}(1+\sqrt{61})$ and asymptotically unstable if $a > \frac{5}{6}(1+\sqrt{61})$.
[You may apply the *derivative test* without proof. Hint: Recall that the eigenvalues of a 2×2 matrix A are given by $\lambda_\pm = \mathrm{tr}(A)/2 \pm \sqrt{(\mathrm{tr}(A)/2)^2 - \det(A)}$, where $\mathrm{tr}(A)$ denotes the trace of A and $\det(A)$ its determinant.]

(b) Show that

(i) The quadrant $\{(x,y) \mid x \geq 0,\ \mathbf{y} \geq 0\}$ is <u>positive flow-invariant</u>.

(ii) All ω-limit sets of the flow are contained in the region

$$B_a := \{(x,y) \mid a \geq x \geq 0,\ 1+a^2 \geq y \geq 0\}.$$

[Hint: consider the flow through the boundary of B_c for all $c \geq a$.]

(iii) Apply the Poincaré-Bendixson Theorem to show that there exists a periodic solution in B_a if $a > \frac{5}{6}(1+\sqrt{61})$, and that this periodic solution must encircle the equilibrium.

(c) Suppose that the equilibrium is the unique ω-limit set of the ODE when $a < \frac{5}{6}(1+\sqrt{61})$. What stability property would you expect for the equilibrium at $a = \frac{5}{6}(1+\sqrt{61})$? Motivate your answer.