Inner products on $\mathbb{C}^m$

On the complex vector space $\mathbb{C}^m$, an inner product is a positive definite conjugate-symmetric sesquilinear (or *Hermitian*) form $\langle\cdot,\cdot\rangle \ : \ \mathbb{C}^m \times \mathbb{C}^m \to \mathbb{C}$:

- positive definite $\langle \mathbf{z},\mathbf{z}\rangle > 0$ if $\mathbf{z} \neq 0$.
- conjugate-symmetric $\langle \mathbf{y},\mathbf{z}\rangle = \overline{\langle \mathbf{z},\mathbf{y}\rangle}$.
- sesquilinear (linear in first component and anti-linear in second component) $\langle a\mathbf{y}, b\mathbf{z}\rangle = a\langle \mathbf{y}, b\mathbf{z}\rangle = \bar{b}\langle a\mathbf{y},\mathbf{z}\rangle$, $a, b \in \mathbb{C}$.

Some remarks:

- We note that if $\mathbf{y},\mathbf{z} \in \mathbb{R}^m$, then $\langle \mathbf{y},\mathbf{z}\rangle$ satisfies the axioms of an inner product on a real vector space (positive-definite symmetric bilinear form).
- The standard (usual) inner product on $\mathbb{C}^m$ is defined as $\langle \mathbf{y},\mathbf{z}\rangle := \sum_{i=1}^m y_i \cdot \overline{z_i}$ where $\mathbf{y} = (y_1 \ldots, y_m)^T$ and $\mathbf{z} = (z_1 \ldots, z_m)^T$. Then $\langle \mathbf{z},\mathbf{z}\rangle = \sum_{i=1}^m |z_i|^2$. Clearly, if $\mathbf{y},\mathbf{z} \in \mathbb{R}^m$ then $\langle \mathbf{y},\mathbf{z}\rangle$ is also the standard inner product on $\mathbb{R}^m$.
- If we identify $\mathbb{C}^m$ with $\mathbb{R}^{2m}$ by $(a_1 + ib_1, \ldots, a_m + ib_m) \simeq (a_1, \ldots, a_m, b_1, \ldots, b_m)$ with $a_i, b_i \in \mathbb{R}$, then the real part of the standard inner product on $\mathbb{C}^m$ is identical to the corresponding standard real inner product on $\mathbb{R}^{2m}$: $\mathrm{Re}(\langle \mathbf{y},\mathbf{z}\rangle_{\mathbb{C}^m}) = \langle \mathbf{y},\mathbf{z}\rangle_{\mathbb{R}^{2m}}$.