# Chapter 5

# The flow near an equilibrium

It is important to study the flow of an ODE. In the absence of general methods that give direct global insight (this absence lies at the basis of what now is known as "chaotic" dynamics), it has proven to be a rather succesful strategy to first understand the nature of a flow in small regions of the phase space and then use this knowledge to obtain more insight into global properties of flows.

For any explicitly given ODE it is typically very hard (if not impossible) to obtain a full understanding of the flow at every level of detail. It may be interesting to point out in this context, that one of *Hilbert's 100 mathematical challenges for the 20th century* (number 16) concerns the question "What is the maximum number of (isolated) periodic solutions that an ODE with polynomial vector field can have (as function of the dimension of phase space and degree of the polynomial)?". Despite lots of effort by many prominent mathematicians, this problem remains unsolved to date.

A more succesful approach (that is at the basis of what is now known as *dynamical systems* theory) concerns the less ambitious aim to understand flows of typical ODEs. This has been a very fruitful and successful approach, that has relied very much on understanding typical local geometric properties of flows.

In this chapter we focus on understanding the nature of flows near equilibria.

## 5.1 Linear approximation

We consider an ODE

$$\frac{d\mathbf{x}}{dt} = f(\mathbf{x}),\ \mathbf{x} \in \mathbb{R}^m, \tag{5.1.1}$$

with equilibirum $\mathbf{x}_0$, i.e. $f(\mathbf{x}_0) = 0$. If $f$ is differentiable (which we will assume here), then we may want to try to approximate $f$ by its first order Taylor expansion near $\mathbf{x}_0$. We consider points $\mathbf{x} + \mathbf{x}_0$ with $|\mathbf{x}|$ small, then using the fact that $f(\mathbf{x}_0) = 0$ we find

$$\frac{d(\mathbf{x}_0 + \mathbf{x})}{dt} = f(\mathbf{x}_0 + \mathbf{x}) = Df(\mathbf{x}_0)\mathbf{x} + O(|\mathbf{x}|^2),$$

from which it follows that the linear ODE

$$\frac{d\mathbf{x}}{dt} = Df(\mathbf{x}_0)\mathbf{x},$$

serves as the *first order* or *linear* approximation of the ODE (5.1.1) near its equilbrium $\mathbf{x}_0$. As this is a linear ODE, we know all about its flow (see Chapter 2). In fact, the first order (linear) approximation of the flow is exactly the flow of the linear approximation of the vector field (see also exercise on problem sheet):

$$D\Phi^t(\mathbf{x}_0) = \exp(Df(\mathbf{x}_0)t).$$

The main question is now to what extent the properties of the linear approximation imply properties for the flow of the nonlinear ODE in the neighbourhood of the equilibrium.

**Definition 5.1.1.**

- We call $A \in gl(m, \mathbb{R})$ hyperbolic if all its eigenvalues lie off the imaginary axis (i.e. have non-zero real part)
- We call an equilibrium $\mathbf{x}_0$ hyperbolic if $Df(\mathbf{x}_0)$ is hyperbolic.

This definition is important, since it turns out that typically matrices are hyperbolic in the following sense:

**Proposition 5.1.2.**

*(a) Suppose that $A \in gl(m, \mathbb{R})$ is not hyperbolic. Then there exists $\delta > 0$ such that $A + \varepsilon I$ (where $I$ denotes the identity matrix) is hyperbolic for all $\varepsilon$ such that $|\varepsilon| \leq \delta$.*

*(b) Suppose that $A \in gl(m, \mathbb{R})$ is hyperbolic. Then there exists $\delta > 0$ such that $A + B$ is hyperbolic for all $B \in gl(m, \mathbb{R})$ satisfying $|B| \leq \delta$.*

*Proof.*

(a) If $A$ is not hyperbolic, then some of the eigenvalues of $A$ lie on the imaginary axis. There are finitely many other eigenvalues that do not lie on the imaginary axis and (since these are only finitely many) there exists a $\delta > 0$ such that the absolute value of the real part of these eigenvalues is always larger than $\delta$. Now observe that if $\lambda$ is an eigenvalue of $A$ then $\lambda + \varepsilon$ is an eigenvalue of $A + \varepsilon I$ (with the same eigenvector). Hence, by the choice of $\delta$, if $0 < |\varepsilon| \leq \delta$ then all eigenvalues of $A + \varepsilon I$ lie off the imaginary axis.

(b) If the matrix $A$ is hyperbolic then there exists a constant $\delta > 0$ such that all eigenvalues of $A$ have a real part with absolute value greater or equal to $\delta$. From the definition of matrix norm, and the triangle inequality it then follows that with $|B| \leq \delta$ the smallest real part of any eigenvalue of the matrix $A + B$ will differ by a magnitude less or equal than $\delta$ from the real part of the eigenvalues of $A$, so that $A+B$ cannot have any eigenvalue on the imaginary axis.

□

Another important point to address is whether equilibria are typically *isolated* in the phase space $\mathbb{R}^m$, or not.

**Example 5.1.3.** The ODE

$$\begin{cases} \dot{x} &= (x-y)^2 \\ \dot{y} &= y-x \end{cases}$$

has a line of equilibria, so the equilibria of this ODE are not isolated.

The following proposition asserts that typically equilibria arise isolated in the phase space.

**Proposition 5.1.4.** *If $\mathbf{x}_0$ is an equilibrium of a vector field $f$ and the Jacobian $Df(\mathbf{x}_0)$ has no eigenvalue equal to zero (i.e. if $Df(\mathbf{x}_0)$ is invertible) then $\mathbf{x}_0$ is an isolated equilibrium.*

**Corollary 5.1.5.** *Typically equilibria of ODEs are isolated.*

*Proof.* Typically the Jacobian of an equilibrium has no zero eigenvalue. □

*Proof of Proposition 5.1.4.* Consider the ODE (5.1.1). We can interpret the equation $f(\mathbf{x}) = 0$ as the intersection of the graph $\mathbf{y} = f(\mathbf{x})$ with $\mathbf{y} = 0$ in the $(\mathbf{x}, \mathbf{y})$-space. Now, if $f(\mathbf{x}_0) = 0$ and $Df(\mathbf{x}_0)$ is invertible, then the tangent spaces to the two graphs $\mathbf{y} = f(\mathbf{x})$ and $\mathbf{y} = 0$ are linearly independent. Namely all tangent vectors to $\mathbf{y} = 0$ are of the form $(\mathbf{x}, 0)$ while the tangent vector to the graph $\mathbf{y} = f(\mathbf{x})$ in $\mathbf{x}_0$ are of the form $(Df(\mathbf{x}_0)\mathbf{y}, \mathbf{y})$. Hence, $Df(\mathbf{x}_0)$ is invertible, these two tangent spaces have trivial intersection $(0, 0)$. So the intersection is transverse. We can apply the formula for the dimension of transverse intersections of "surfaces": if in $\mathbb{R}^p$ a surface of dimension $q$ transversely intersects a surface of dimension $r$ then (locally, near the intersection point) the intersection is a surface of dimension $p - q - r$ (see problem sheet 3, exercise 8). We obtain in this case that the dimension is $2m - m - m = 0$ (where $m$ denotes the dimension of the phase space of the ODE). By application of the implicit function theorem, transverse intersections are *persistent* under small (smooth) perturbations of the surfaces). □

**Remark 5.1.6.** Formally, the notion of *small perturbation* needs some agreement on when two vector fields are close. Without going into further detail, we here assume small in the context of the so-called *(Whitney) smooth topology*: two $C^k$ functions (here vector fields) are close if the functions, and all their partial derivatives (up to order $k$) are close at all points in the domain of definition.

**Example 5.1.7.** The following perturbation ($|\varepsilon| << 1$) of the ODE in Example 5.1.3

$$\begin{cases} \dot{x} &= (x-y)^2 \\ \dot{y} &= y-x+\varepsilon x^2 \end{cases}$$

has only one (isolated) equilibrium $x = y = 0$ if $\varepsilon \neq 0$. So the situation.

## 5.2 Hyperbolic equilibria

Typically (e.g. if equilibria are hyperbolic) equilibria are isolated in the phase space . The question remains what the flow is near such an isolated equilibrium? We like to compare it to the linear approximation of the flow: let $\mathbf{x}_0$ be an equilibrium of a flow $\Phi^t$ and $\mathbf{y} := \mathbf{x} - \mathbf{x}_0$, then this linear approximation is given by $\mathbf{y}(t) = D\phi^t(\mathbf{x}_0))\mathbf{y}_0$ (with $\mathbf{y}(0) = \mathbf{y}_0$). In particular if $\mathbf{y} = 0$ then $\mathbf{x} = \mathbf{x}_0$ and $\mathbf{y}(t) = 0\ \forall t$ or equivalenty that $\mathbf{x}(t) = \mathbf{x}_0\ \forall t$.

Recall that $D\Phi^t(\mathbf{x}_0)\exp(Df(\mathbf{x}_0))$ is $\Phi^t$ is the flow of the vector field $f$. We consider some different situations:

1. **All eigenvalues of $Df(\mathbf{x}_0)$ have negative real part.** Then, for $t > 0$ the eigenvalues of $D\Phi^t(\mathbf{x}_0)$ have absolute value small than 1. By continuity of the derivative (we assume that $f$ is $C^1$) it follows that this property extends to some small neighborhood of $\mathbf{x}_0$: all eigenvalues of $Df(\mathbf{x})$ have negative real part if $|\mathbf{x} - \mathbf{x}_0|$ is sufficiently small. Now we can apply the derivative test. If for all $\mathbf{x}$ in a closed ball $\overline{B(\mathbf{x}_0, \varepsilon)}$ the derivative $D\Phi^t(\mathbf{x})$ $(t > 0)$ has all eigenvalues inside the unit circle so that $\Phi^t$ $(t > 0)$ is a contraction on $\overline{B(\mathbf{x}_0, \varepsilon)}$. Hence, $\Phi^t$ has a unique fixed point in $\overline{B(\mathbf{x}_0, \varepsilon)}$ and all initial conditions in this neighbourhood converge exponentially fast to $\mathbf{x}_0$ as $t \to \infty$. The equilibrium $\mathbf{x}_0$ is asymptotically stable. We call $x_0$ a *sink* or more generally an *attractor*. The region $\overline{B(\mathbf{x}_0, \varepsilon)}$ is part of the *basin of attraction* of $\mathbf{x}_0$, which is defined as the set of all initial conditions that converge to $\mathbf{x}_0$ as $t \to \infty$.

2. **All eigenvalues of $Df(\mathbf{x}_0)$ have positive real part.** Then, for $t > 0$ the eigenvalues of $D\Phi^t(\mathbf{x}_0)$ have absolute value greater than 1, and all eigenvalues of $D\Phi^t(\mathbf{x})$ have absolute value smaller than 1 if $|\mathbf{x} - \mathbf{x}_0|$ is sufficiently small. Now we can apply the derivative test to obtain that $\Phi^t$ is a contraction if $t < 0$ (!). Hence, $\Phi^t$ has a unique fixed point in $\overline{B(\mathbf{x}_0, \varepsilon)}$ and all initial conditions in this neighbourhood converge exponentially fast to $\mathbf{x}_0$ as $t \to -\infty$: the equilibrium $\mathbf{x}_0$ is asymptotically unstable. We call $\mathbf{x}_0$ a *source* or more generally an *repeller*.

3. **All eigenvalues of $Df(\mathbf{x}_0)$ have nonzero real part and there exist eigenvalues with positive as well as negative real part.** This situation is more complicated and we discuss the situation in more detail in the remainder of this section.

**Example 5.2.1.** Consider the linear ODE in the plane

$$\begin{cases} \dot{x} &= x \\ \dot{y} &= -y. \end{cases}$$

The origin 0 is an equilibrium and $Df(0)$ has eigenvalues $\pm 1$. We have seen before that there is a stable subspace $E^s := \{x = 0\}$ for which $\forall \mathbf{x} \in E^s$ and $t \geq 0$ we have $\Phi^t(\mathbf{x}) \in E^s$ and $\lim_{t\to\infty} \Phi^t(\mathbf{x}) = 0$, and an unstable subspace $E^u := \{y = 0\}$ for which $\forall \mathbf{x} \in E^u$ and $t \leq 0$ we have $\Phi^t(\mathbf{x}) \in E^u$ and $\lim_{t\to-\infty} \Phi^t(\mathbf{x}) = 0$.

In this example we observe *mixed* behaviour: on $E^s$ we have exponential attraction to the equilibrium and on $E^u$ we have exponential repulsion from the equilibrium. All other initial conditions show potentially mixed behaviour between being attracted and repelled from the equilibrium (and all such solutions go to infinity if $t \to \pm\infty$). Note that by continuity of the flow (for any finite $t$), if we start very close to the stable subspace, for a very long time it looks as if the solution tends to the equilibrium. Only when it comes very close the equilibrium the orbit suddenly escapes along the direction of the unstable subspace. If we measure the time it takes to go from $(x, y_0)$ to the line $x = x_0$ then this time tends to infinity if $x \to 0$. The question is whether this kind of behavior is observed also in the case of nonlinear ODEs.

**Theorem 5.2.2** (Hartman-Grobman)**.** *There exists a continuous change of coordinates such that in the neighbourhood of a hyperbolic equilibrium the flow becomes linear in this neighbourhood.*

In other words, there exists $T$ invertible and continuous such that such that $T \circ \Phi^t \circ T^{-1} = D\Phi^t(\mathbf{x}_0)$ near the equilibrium $\mathbf{x}_0$. We do not prove this theorem here, but it can be proven using the contraction mapping theorem.

**Example 5.2.3.** Consider the planar ODE

$$\left\{ \begin{array}{rcl} \dot{x} &=& x + y^2 \\ \dot{y} &=& -y. \end{array} \right. \tag{5.2.1}$$

which has the origin as an equilibrium point. The derivative $Df(0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ satisfies the condition of the Hartman-Grobman theorem and thus we know that near the equilibrium point one can choose coordinates such that the flows is that of the linear vector field $Df(0)$ (which is determined by the flow on the one-dimensional stable and unstable subspaces).

In this example we can actually solve the initial value problem for the ODE (5.2.1) with $(x(0), y(0)) = (x_0, y_0)$. Namely

$$\begin{array}{rcl} \dot{y} &=& -y \quad \Rightarrow \quad y(t) = y_0 e^{-t}, \\ \dot{x} &=& x + y^2 = x + y_0^2 e^{-2t} \quad \Rightarrow \quad x(t) = (x_0 + \frac{1}{3} y_0^2) e^t - \frac{1}{3} y_0^2 e^{-2t}. \end{array}$$

We see that if $y_0 = 0$ we have $y(t) = 0 \ \forall t$, so that $x$-axis is *flow-invariant.* (Note that this can also be seen directly from the ODE (5.2.1): if $y = 0$ we have $\dot{y} = 0$ and $\dot{x} = x$. Similarly, but less obviously, if the initial conditions satisfy $x_0 = -\frac{1}{3} y_0^2$ then it turns out from the explicit expression of the solution ($x(t) = -\frac{1}{3} y_0^2 e^{-2t}$ and $y(t) = y_0 e^{-t}$) that $x(t) = -\frac{1}{3} y(t)^2 \ \forall t$.

We thus found two flow-invariant curves: $y = 0$ and $x = -\frac{1}{3} y^2$. Moreover, the behaviour on each of these curves is rather similar then on the stable and unstable subspaces of the linear ODE $\dot{x} = x$, $\dot{y} = -y$. Namely, when $y_0 = 0$ we have $x(t) = x_0 e^t$ and the flow is exponentially expanding and $\lim_{t \to -\infty} x(t) = 0$. If $x_0 = -\frac{1}{3} y_0^2$ we observe that

$$|\mathbf{x}(t)| = \sqrt{x(t)^2 + y(t)^2} = |y_0| e^{-t} \sqrt{\frac{1}{9} y_0^2 e^{-2t} + 1} \leq C e^{-t} \quad t \geq 0.$$

for some constant $C$ (e.g. $C = |y_0|\sqrt{\frac{1}{9}y_0^2+1}$). This means that on the curve $x = -\frac{1}{3}y^2$ the flow is exponentially contracting and all initial conditions on this line flow to the equilibrium as $t \to \infty$.

The flow on these invariant curves thus shows similarities to the flow on stable and unstable subspaces of linear ODEs. They are known as *stable and unstable manifolds* and are the nonlinear equivalents of the stable and unstable subspaces of equilibria of linear ODEs. We note that we here use the technical term *manifold*, which you should interpret as "smooth surface" (which is here in fact a curve).

In fact, in the spirit of the Hartman-Grobman theorem, in this (very special!) case there is a smooth global change of coordinates such that the flow becomes linear. Namely, if

$$u = x + \frac{1}{3}y^2, \quad v = y,$$

(note that this is an invertible coordinate transformation) the unstable manifold corresponds to $v = 0$ and the stable manifold to $u = 0$. In terms of these coordinates the ODE has the form

$$\begin{cases} \dot{u} = \dot{x} + \frac{2}{3}y\dot{y} = x + \frac{1}{3}y^2 & = u \\ \dot{v} = \dot{y} = -y & = -v. \end{cases}$$

We now formalise the notion of stable and unstable manifold that we introduced in the above example.

**Definition 5.2.4.** Let $\mathbf{x}_0$ be an equilibrium of an (autonomous) ODE, then the *stable manifold* of $\mathbf{x}_0$ is defined as

$$W^s(\mathbf{x}_0) = \{\mathbf{x} \mid \lim_{t\to\infty} \Phi^t(\mathbf{x}) = x_0\}.$$

Similarly the *unstable manifold* is defined as the set

$$W^u(\mathbf{x}_0) = \{\mathbf{x} \mid \lim_{t\to-\infty} \Phi^t(\mathbf{x}) = x_0\}.$$

We have the following theorem (which can be considered as a corollary of he Hartman-Grobman Theorem):

**Theorem 5.2.5.** *If $\mathbf{x}_0$ is a hyperbolic equilibrium of an ODE $\dot{\mathbf{x}} = f(\mathbf{x})$, then there exist stable and unstable manifolds $W^s(\mathbf{x}_0)$ and $W^u(\mathbf{x}_0)$ that are tangent to the stable and unstable subspaces $E^s$ and $E^u$ of the linear ODE $\dot{\mathbf{y}} = Df(\mathbf{x}_0)\mathbf{y}$ (with $\mathbf{y} = \mathbf{x} - \mathbf{x}_0$) so that under the flow of the ODE every point in $W^s(\mathbf{x}_0)$ converges with exponential speed to $\mathbf{x}_0$ as $t \to \infty$ and every point in $W^u(\mathbf{x}_0)$ converges with exponential speed to $\mathbf{x}_0$ as $t \to -infty$.*

We do not prove the existence or smoothness of these manifolds here, but this result can be proven using the Contraction Mapping Theorem.