UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)
MAY–JUNE 2007

This paper is also taken for the relevant examination for the Associateship.

M2A3 One-Dimensional Fluid Mechanics

Date: Thursday, 17th May 2007 Time: 10 am – 12 pm

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. Fluid flows through a pipe of constant rectangular cross-section A. The fluid velocity at a location x along the pipe is given by $u(x,t)$ and the fluid density is $\rho(x,t)$.

(i) Define mathematically the mass flux $Q(x,t)$ and momentum flux $F(x,t)$ and explain what each represents physically.

In addition to the force due to the pressure $p(x,t)$, a force per unit volume $R(x,t)$ acts on the fluid in the $x-$direction.

(ii) By concentrating on the fluid motion in a fixed region $a \leq x \leq b$, derive the governing equations

$$A\frac{d}{dt}\int_a^b \rho\, dx = Q(a,t) - Q(b,t),$$

$$A\frac{d}{dt}\int_a^b \rho\, u\, dx = F(a,t) - F(b,t) + A\left(p(a,t) - p(b,t)\right) + A\int_a^b R(x,t)\, dx.$$

(iii) Show that these equations simplify to

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0, \quad \rho\frac{Du}{Dt} = -\frac{\partial p}{\partial x} + R,$$

where D/Dt is the material derivative. State clearly any assumptions you have made in deriving this final pair of equations.

2. Fluid of constant density ρ_0 and depth $h(x)$ flows steadily in the $x-$direction through an open horizontal channel of cross-sectional area $A(h)$. The governing equations are

$$\frac{d}{dx}(A\,u) = 0, \;\; u\frac{du}{dx} = -g\frac{dh}{dx},$$

where $u(x)$ is the fluid velocity and g is the acceleration due to gravity.

(i) Show that the volume flux Q is given by

$$Q^2 = 2g(E-h)A^2,$$

where E is a constant.

The channel is bounded by a parabola so that the cross-sectional region occupied by the fluid is

$$az^2 \leq y \leq h,$$

where y & z are, respectively, the vertical and horizontal coordinates in the cross-section, and a is a constant.

(ii) Show that $A(h) = 4h^{3/2}/(3a^{1/2})$.

(iii) Find the maximum value of Q for fixed E, and determine the corresponding values of fluid velocity and depth in terms of E.

(iv) Hence or otherwise, deduce the speed at which small unsteady perturbations will propagate along this channel, relative to a uniform base flow u_0.

3. The equations of motion for an unsteady one-dimensional open channel flow with free surface $h(x,t)$ and rectangular cross-section are, in the usual notation:

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(h\,u) = 0, \quad \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -g\frac{\partial h}{\partial x}.$$

(i) Assuming a height-velocity relation of the form $h = h(u)$, show that solutions to the above equations are only possible if

$$h = g\,(h')^2\,.$$

(ii) Solve this equation for h, applying $h = h_0$ when $u = 0$. Assuming that the height is a decreasing function of velocity, show that $u(x,t)$ satisfies the kinematic wave equation

$$\frac{\partial u}{\partial t} + \left(\frac{3}{2}u - \sqrt{gh_0}\right)\frac{\partial u}{\partial x} = 0.$$

Deduce that u is constant along the straight-line characteristics $dx/dt = R(x,t)$, and identify the function R.

Suppose that at time $t = 0$ the velocity is given by

$$u(x,0) = \begin{cases} 0 & x \leq 0, \\ -U_0\,(x/L) & 0 \leq x \leq L, \\ -U_0 & x \geq L, \end{cases}$$

where U_0 is a positive constant.

(iii) On an $x - t$ plot, sketch the individual characteristics that originate from $t = 0$ at the following values of x :

$$0,\ L/2,\ L.$$

Show that these characteristics intersect when $t = 2L/(3U_0)$ and find the corresponding value of x. What can be deduced about the nature of the solution at this time?

4. Constant-density fluid flows along an elastic-walled pipe of circular cross-section in which the radius $r(x,t)$ is related to the pressure $p(x,t)$ by

$$r^2 = r_0^2 + \alpha^2 (p - p_0),$$

where r_0, α, p_0 are constants and x is measured along the pipe axis.

(i) Starting from the unsteady equations for one-dimensional flow through a vessel of arbitrary cross-section, derive the governing equations

$$\frac{\partial}{\partial t}(r^2) + \frac{\partial}{\partial x}(r^2 u) = 0, \quad \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{1}{\rho_0 \alpha^2}\frac{\partial}{\partial x}(r^2).$$

(ii) Suppose that the flow is perturbed slightly from the uniform state $u = u_0, r = r_0$. Show that the velocity perturbation $\widetilde{u}(x,t)$ satisfies the convected wave equation, and deduce that it propagates with speeds $u_0 \pm c_0$, where $c_0 = r_0 / \left(\alpha \rho_0^{1/2}\right)$.

(iii) Guided by the linearized result of part (ii) or otherwise, define a suitable nonlinear wavespeed $c(r)$, and show that the nonlinear equations of part (i) can be written in the alternative form

$$\left(\frac{\partial}{\partial t} + (u+c)\frac{\partial}{\partial x}\right)(u + 2c) = 0, \quad \left(\frac{\partial}{\partial t} + (u-c)\frac{\partial}{\partial x}\right)(u - 2c) = 0.$$

How do these equations compare with those for flow through an open channel of rectangular cross-section?

5. An isentropic gas evolves according to the equations

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0, \quad \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -k^2 \rho \frac{\partial \rho}{\partial x},$$

where k and γ are constants.

(i) Show that the equations may be written in the characteristic form

$$\left(\frac{\partial}{\partial t} + (u+a)\frac{\partial}{\partial x}\right)(u+a) = 0, \quad \left(\frac{\partial}{\partial t} + (u-a)\frac{\partial}{\partial x}\right)(u-a) = 0, \qquad (1)$$

where $a = k\rho$.

(ii) Suppose that at $t = 0$ the gas is at rest at constant density ρ_0 with $a = a_0$. Deduce that equations (1) admit a solution with $(u-a)$ constant everywhere, and u constant along the straight-line characteristics

$$\frac{dx}{dt} = 2u + a_0.$$

What is the equation of the characteristic that passes through $x = 0, t = 0$?

Initially the gas occupies the region $x > 0$, and there is a rigid wall at $x = 0$. From time $t = 0$ onwards the wall moves in the positive $x-$direction and its position is denoted by $X(t)$.

(iii) Assuming a solution of the form given in (ii), and by considering a characteristic originating from the wall at time τ, deduce that in the region $X(t) \le x \le a_0 t$:

$$u(x,t) = X'(\tau)$$

with τ given in terms of x and t by

$$x - X(\tau) = (2X'(\tau) + a_0)(t - \tau).$$

Find the corresponding solution for $a(x,t)$.

(iv) What are the solutions for u and a in the region $x > a_0 t$?