Imperial College London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2006

This paper is also taken for the relevant examination for the Associateship.

M2A3

One-Dimensional Fluid Mechanics

Date: Friday, 19th May 2006 Time: 10 am – 12 noon

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. (i) Give a brief outline of the Eulerian and Lagrangian descriptions of unsteady one-dimensional fluid motion and explain how the material derivative D/Dt is defined.

(ii) Fluid flows through a pipe with constant rectangular cross-sectional area A. The fluid velocity at a location x along the pipe is denoted by $u(x,t)$ and the fluid density by $\rho(x,t)$. By concentrating on the fluid motion in a fixed region $a \leq x \leq b$, use the concepts of mass and momentum flux to derive the governing equations

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) &= 0, \\ \frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho u^2) &= -\frac{\partial p}{\partial x}. \end{aligned}$$

You may assume that the only force acting on the fluid is due to the pressure $p(x,t)$. Show that the second of these equations can be written in the alternative form

$$\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x}.$$

(iii) Explain how the above equations are modified if, rather than through an enclosed pipe, the flow is along an open horizontal channel of rectangular cross-section with constant width, but variable depth $h(x,t)$, and the fluid is of constant density.

2. In an elastic-walled pipe of circular cross-section, the radius $r(x,t)$ and fluid velocity $u(x,t)$ are governed by the equations

$$\frac{\partial}{\partial t}(r^2) + \frac{\partial}{\partial x}(r^2 u) = 0, \quad \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{2}{\alpha^2}\frac{\partial r}{\partial x}, \tag{1}$$

where α is a constant.

Show that small perturbations to the rest state $u = 0, r = r_0$ (r_0 constant) propagate with speed $c_0 = r_0^{1/2}/\alpha$.

Now consider the steady version of (1), and show that upon integration r satisfies:

$$Q^2 = \left(\frac{2}{\alpha}\right)^2 r^4 (E - r),$$

where Q and E are constants. Interpret the constant Q physically.

Show that Q is a maximum ($Q_{\max}$, say) when $r = 4E/5$.

By sketching Q^2 versus r, show graphically that, for a given value of Q ($< Q_{\max}$) there are two possible values of r. What are the corresponding values of u?

By defining an appropriate Froude number, show that one of these flows is subcritical and the other supercritical.

3. The one-dimensional flow of a river of depth $h(x,t)$ can be modelled by the nonlinear equations

$$\frac{\partial h}{\partial t}+\frac{\partial}{\partial x}(hu)=0,$$
$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=-g\frac{\partial h}{\partial x}\cos\alpha+g\sin\alpha-D_0\frac{u^2}{h}.$$

Here, α is the slope of the river and the constant D_0 is a drag coefficient.

(i) Show that there is a uniform solution with $u=u_0, h=h_0$ and $u_0^2=(g\sin\alpha/D_0)h_0$.

(ii) Seek small perturbations to this basic state by writing

$$u=u_0+\widetilde{u}(x,t),\quad h=h_0+\widetilde{h}(x,t),$$

and show that $\widetilde{u}$ and $\widetilde{h}$ satisfy the linearised equations

$$\left(\frac{\partial}{\partial t}+u_0\frac{\partial}{\partial x}\right)\widetilde{h}=-h_0\frac{\partial\widetilde{u}}{\partial x},$$
$$\left(\frac{\partial}{\partial t}+u_0\frac{\partial}{\partial x}\right)\widetilde{u}=-g\frac{\partial\widetilde{h}}{\partial x}\cos\alpha-D_0\frac{u_0^2}{h_0}\left(\frac{2\widetilde{u}}{u_0}-\frac{\widetilde{h}}{h_0}\right).$$

(iii) Show that, upon elimination of $\widetilde{u}$, the perturbation to the depth satisfies

$$\left(\frac{\partial}{\partial t}+u_0\frac{\partial}{\partial x}\right)^2\widetilde{h}=c_0^2\frac{\partial^2\widetilde{h}}{\partial x^2}-g\sin\alpha\left(\frac{2}{u_0}\frac{\partial\widetilde{h}}{\partial t}+3\frac{\partial\widetilde{h}}{\partial x}\right),$$

where $c_0^2=gh_0\cos\alpha$.

(iv) Seek wave-like solutions for $\widetilde{h}$ proportional to $\exp\left[ik(x-\widetilde{c}t)\right]$, with k real, and derive the dispersion relation

$$\left(3-\frac{2\widetilde{c}}{u_0}\right)g\sin\alpha+ik\left((u_0-\widetilde{c})^2-c_0^2\right)=0.$$

Consider a solution for $\widetilde{c}$, valid for small k, of the form

$$\widetilde{c}=c_1+kc_2+\cdots.$$

Show that $c_1=3u_0/2$ and find c_2.

Deduce that long wavelength disturbances on the river will grow exponentially in time provided $4D_0<\tan\alpha$.

4. Constant-density fluid of depth h flows along an open channel. The governing equations are

$$\frac{\partial(h^2)}{\partial t} + \frac{\partial}{\partial x}(h^2 u) = 0, \ \ \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -g\frac{\partial h}{\partial x}.$$

(i) Show that these equations may be written in the alternative form

$$\left(\frac{\partial}{\partial t} + (u+c)\frac{\partial}{\partial x}\right)(u+4c) = 0, \ \ \left(\frac{\partial}{\partial t} + (u-c)\frac{\partial}{\partial x}\right)(u-4c) = 0, \qquad (1)$$

where $c^2 = \frac{1}{2}gh$.

(ii) Suppose that $u = 0$ when $h = h_0$ (constant). Deduce that equations (1) admit a solution with $(u - 4c)$ constant everywhere, and u constant along the straight-line characteristics

$$\frac{dx}{dt} = \frac{5}{4}u + c_0, \ \ c_0^2 = \frac{1}{2}gh_0.$$

The channel is bounded at $x = 0$ by a wall, with the fluid occupying the region $x > 0$. Initially the fluid is at rest and has depth h_0. From time $t = 0$ onwards the wall moves in the negative $x-$direction with speed αt (α constant).

(iii) Assuming a solution of the form given in (ii), deduce that in the region $x \geq c_0 t$, the flow remains in its initial state.

(iv) By considering a characteristic originating from the wall at time τ, show that in the region $-\frac{1}{2}\alpha t^2 \leq x \leq c_0 t$ the solution for u is

$$u(x,t) = -\alpha\tau(x,t),$$

with τ given in terms of x and t by

$$x + \frac{1}{2}\alpha\tau^2 = \left(c_0 - \frac{5}{4}\alpha\tau\right)(t-\tau).$$

Find the corresponding solution for $h(x,t)$.

5. An isentropic gas evolves according to the equations

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0, \quad \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\rho\frac{\partial \rho}{\partial x}, \tag{1}$$

in the usual notation.

(i) Show that small perturbations $\widetilde{u}(x,t), \widetilde{\rho}(x,t)$ to the uniform state $u = u_0, \rho = \rho_0$ satisfy a convected wave equation, and deduce that the wave disturbances propagate at speeds $u_0 \pm \rho_0$.

(ii) Show that the nonlinear equations (1) can be rewritten in the form

$$\left(\frac{\partial}{\partial t} + (u+\rho)\frac{\partial}{\partial x}\right)(u+\rho) = 0, \quad \left(\frac{\partial}{\partial t} + (u-\rho)\frac{\partial}{\partial x}\right)(u-\rho) = 0.$$

Deduce that $(u+\rho), (u-\rho)$ are constant along straight-line characteristics, and write down the equations of these characteristics.

(iii) Hence or otherwise deduce that the general solution for u can be written as

$$u = F(x-(u+\rho)t) + G\left(x-(u-\rho)t\right),$$

where F, G are arbitrary functions. Find the corresponding solution for ρ. What is the relationship between u and ρ that ensures that waves propagate in both the positive and negative $x-$directions?