Imperial College London

UNIVERSITY OF LONDON

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BSc and MSci EXAMINATIONS (MATHEMATICS)
MAY–JUNE 2005

This paper is also taken for the relevant examination for the Associateship.

M2A3 ONE-DIMENSIONAL FLUID DYNAMICS

Date: Thursday, 19th May 2005 Time: 2 pm – 4 pm

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. An isentropic gas evolves according to the equations

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) &= 0, \\ \rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} \right) &= -k^2 \frac{\partial \rho}{\partial x}, \end{aligned}$$

where u is the velocity, ρ the density, t the time, x the position, and k is a constant.

(i) Assuming a density-velocity relation of the form $\rho = \rho(u)$, show that solutions to the above equations are only possible if

$$\rho = \pm k \rho'(u).$$

(ii) Solve this equation for ρ, applying the condition $\rho = \rho_0$ (constant) when $u = 0$. Assuming that the density is a decreasing function of velocity, show that $u(x,t)$ satisfies the kinematic wave equation

$$\frac{\partial u}{\partial t} + (u - k) \frac{\partial u}{\partial x} = 0, \tag{1}$$

and deduce the speed of propagation of wave disturbances as a function of velocity.

(iii) Suppose that at time $t = 0$ the density distribution is of the form

$$\rho(x, 0) = \begin{cases} \frac{1}{2}\rho_0 & x < 0, \\ \rho_0 & x > 0, \end{cases}$$

with a shock at $x = 0$. By considering the kinematic wave equation (1) in conservative form, show that by time t the shock front has advanced a distance

$$k \left(1 - \frac{1}{2} \ln 2 \right) t$$

to the left.

2. A stream of water of depth h_1 and speed u_1 undergoes a hydraulic jump. The depth and speed of the fluid to the right of the jump are h_2 and u_2, with $h_2 > h_1$. Acceleration due to gravity is denoted by g.

(i) Derive the relations

$$u_1 h_1 = u_2 h_2,$$
$$h_1 u_1^2 + \frac{1}{2} g h_1^2 = h_2 u_2^2 + \frac{1}{2} g h_2^2,$$

linking the properties upstream and downstream of the jump.

(ii) Using the equations in part (i), obtain expressions for the upstream and downstream Froude numbers F_1 and F_2 in terms of h_1 and h_2, and show that

$$F_1^2 = \left(\frac{h_2}{h_1}\right)^3 F_2^2.$$

(iii) Deduce that $F_1 > 1$ and $F_2 < 1$.

3. A fluid of constant density ρ_0 and depth h flows steadily through an open horizontal channel of cross-sectional area $A = ah^2 + bh$, with a and b positive constants. The governing equations are

$$u\frac{\mathrm{d}u}{\mathrm{d}x} = -g\frac{\mathrm{d}h}{\mathrm{d}x}, \quad \frac{\mathrm{d}}{\mathrm{d}x}(Au) = 0,$$

where u is the fluid velocity and g is acceleration due to gravity.

(i) Draw a possible shape for the cross-section.

(ii) Show that the flux Q is related to h by the equation

$$Q^2 = 2g(E - h)(ah^2 + bh)^2,$$

where E is a constant.

(iii) Show that for fixed E, $\mathrm{d}Q^2/\mathrm{d}h = 0$ provided h satisfies

$$5ah^2 + (3b - 4aE)h - 2Eb = 0.$$

Show that this equation has two real roots of opposite sign. Deduce that the flux is a maximum ($Q_{\max}$, say) when h is the positive root of the quadratic.

(iv) Sketch Q^2 versus h for fixed E and show graphically that there are two flows that provide a given flux Q ($< Q_{\max}$).

(v) Define the Froude number Fr in the usual way, and show that for $a = 0$, one of these flows has Fr < 1, while the other has Fr > 1.

4. **(i)** Show that the convected wave equation for $h(x,t)$:

$$\left(\frac{\partial}{\partial t} + U_0 \frac{\partial}{\partial x}\right)^2 h = c_0^2 \frac{\partial^2 h}{\partial x^2}, \tag{1}$$

with U_0, c_0 constant, can be transformed into the standard wave equation:

$$\frac{\partial^2 h}{\partial t^2} = c_0^2 \frac{\partial^2 h}{\partial \xi^2}, \tag{2}$$

where $\xi = x - U_0 t$. Hence deduce that the general solution of (1) is

$$h(x,t) = F\left(x - (U_0 + c_0)t\right) + G\left(x - (U_0 - c_0)t\right),$$

where F and G are arbitrary functions. [You may quote the general solution of (2)].

(ii) A fluid flows with constant speed U_0 and depth h_0 in a channel of rectangular cross-section. At time $t = 0$ the surface is disturbed slightly, so that

$$h(x,0) = h_0 + \varepsilon \mathrm{e}^{-kx^2}, \quad \frac{\partial h}{\partial t}(x,0) = 0,$$

with $0 < \varepsilon \ll h_0$. Given that h satisfies the convected wave equation, find the solution for $h(x,t)$ for $t > 0$.

(iii) Define the Froude number Fr of the undisturbed flow and sketch the solution for h at some time $\tau > 0$ in each of the following cases:

(a) Fr < 1, (b) Fr $= 1$, (c) Fr > 1.

5. Water flows through an elastic-walled pipe of circular cross-section in which the radius $r(x,t)$ is related to the pressure $p(x,t)$ by

$$r = r_0 + \alpha(p - p_0),$$

where r_0, p_0 and α are positive constants. The coordinate x is measured along the pipe axis.

(i) Starting from the unsteady equations for one-dimensional flow through a vessel of arbitrary cross-section, and assuming that the fluid motion is parallel to the pipe axis for all x and t, show that the governing equations for the fluid motion are

$$\frac{\partial}{\partial t}(r^2) + \frac{\partial}{\partial x}(r^2 u) = 0,$$
$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{1}{\rho_0 \alpha}\frac{\partial r}{\partial x},$$

where $u(x,t)$ is the fluid velocity and ρ_0 is the constant density.

(ii) Suppose that the flow is perturbed slightly from its rest state where $u = 0$ and $r = r_0$. Show that the velocity perturbation $\widetilde{u}(x,t)$ satisfies the wave equation

$$\frac{\partial^2 \widetilde{u}}{\partial t^2} = c_0^2 \frac{\partial^2 \widetilde{u}}{\partial x^2},$$

with $c_0^2 = r_0/2\rho_0\alpha$, and find the corresponding equation for $\widetilde{r}$, the perturbation to $r = r_0$.

(iii) Guided by the result of part (ii), or otherwise, define a suitable nonlinear wavespeed $c(r)$, and show that the nonlinear equations derived in part (i) can be rewritten in the form

$$\left(\frac{\partial}{\partial t} + (u+c)\frac{\partial}{\partial x}\right)(u + 4c) = 0,$$
$$\left(\frac{\partial}{\partial t} + (u-c)\frac{\partial}{\partial x}\right)(u - 4c) = 0.$$

Deduce that $u + 4c$ and $u - 4c$ are constant along characteristic curves, and write down the differential equations for these curves.