Imperial College London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)
MAY–JUNE 2004

This paper is also taken for the relevant examination for the Associateship.

M2A3 ONE-DIMENSIONAL FLUID DYNAMICS

Date: Wednesday, 19th May 2004 Time: 2 pm – 4 pm

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. **(i)** Give a brief outline of the Eulerian and Lagrangian descriptions of one-dimensional unsteady fluid motion, and explain how the material derivative D/Dt is defined.

(ii) The Eulerian velocity for a fluid flow is given by

$$u^E(x,t) = \frac{3t^2 x}{1+t^3} + 1 + t^3.$$

A fluid particle is at the location $x = \xi$ at time $t = 0$. Find its position and acceleration at time t in terms of ξ and t. Verify that the particle acceleration is equal to Du^E/Dt.

(iii) Explain briefly the meaning of the following:

- **(a)** Froude number
- **(b)** Hydraulic jump
- **(c)** Bore
- **(d)** Mach number
- **(e)** Shock wave

2. An isentropic gas evolves according to the equations

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) &= 0, \\
\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) &= -k^2 \rho \frac{\partial \rho}{\partial x},
\end{aligned}
$$

where u is the velocity, ρ the density, x distance, t time and k is a constant.

(i) Assuming a density-velocity relation of the form $\rho = \rho(u)$, show that solutions to the above equations are only possible if

$$\rho = k^2(\rho'(u))^2.$$

(ii) Solve this equation for ρ, applying the condition $\rho = \rho_0$ (constant) when $u = 0$. Assuming that the density is an increasing function of velocity, show that $u(x,t)$ satisfies a kinematic wave equation of the form

$$\frac{\partial u}{\partial t} + \left(\frac{3}{2}u + k\rho_0^{1/2}\right)\frac{\partial u}{\partial x} = 0. \tag{1}$$

What is the speed of propagation of the waves as a function of u?

Consider an initial velocity distribution of the form

$$u(x,0) = \begin{cases} U_0, & x < 0, \\ 0, & x > 0, \end{cases} \qquad (U_0 \text{ constant}),$$

so that there is a shock at $x = 0$ separating two regions of uniform velocity.

(iii) Verify that the characteristics of (1), emanating from $t = 0$, cross in the first quadrant of the $x - t$ plane. What are the implications of this event?

(iv) By considering the kinematic wave equation in conservative form, show that at time t the shock front has advanced a distance

$$\left(\frac{3}{4}U_0 + k\rho_0^{1/2}\right)t.$$

3. **(a)** Fluid of constant density ρ_0 flows at speed u through a pipe with cross-sectional area $A(x,t)$ where x is the direction of motion of the fluid and t is time. The pipe is inclined at an angle α to the horizontal, and the equations of motion are:

$$\rho_0 \frac{\partial A}{\partial t} + \rho_0 \frac{\partial}{\partial x}(Au) = 0,$$

$$\rho_0 \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} \right) = -\frac{\partial p}{\partial x} + \rho_0 g \sin\alpha,$$

$$0 = -\frac{\partial p}{\partial z} - \rho_0 g \cos\alpha,$$

with g the acceleration due to gravity, p pressure and z measured normal to the slope.

(i) Explain briefly the fundamental principles involved in the derivation of these equations and give a physical interpretation of each term.

(ii) How do these equations simplify when the flow is steady?

(b) A tap supplies a constant flux Q_0 of water. The water flows out of the tap vertically downwards in the form of a jet at constant atmospheric pressure p_0. The jet has a circular cross-section with radius $r(y)$ where y is measured in the direction of gravity, and $r = r_0$ when $y = y_0$.

(i) Show that

$$r = r_0 \left(\frac{2\pi^2 r_0^4 g}{Q_0^2}(y - y_0) + 1 \right)^{-1/4}.$$

(ii) In addition to the information given in the question, what extra assumption is made in obtaining this solution? Show that $dr/dy \to 0$ as $y \to \infty$ and discuss the relevance of this result to the assumption you stated.

4. A fluid of constant density ρ_0 flows steadily through a horizontal channel with triangular cross-section $A = h^2 \tan\alpha$, where α is a constant.

(i) Use Bernoulli's equation and the equation of continuity to show that the flux Q is related to the fluid depth h by the equation

$$Q^2 = 2h^4(E-h)g\tan^2\alpha,$$

where E is a constant.

(ii) Show that for fixed E, the flux is a maximum when $h = 4E/5$. Find the maximum flux $Q_{\max}$ in terms of E and the corresponding fluid velocity.

(iii) Sketch Q^2 versus h for fixed E and show graphically that there are two flows (u_1, h_1) and (u_2, h_2) that provide the same flux Q $(< Q_{\max})$.

(iv) Demonstrate that one of these flows has a Froude number F (based on h) greater than $1/\sqrt{2}$, and the other has $F < 1/\sqrt{2}$.

5. **(a)** The equations of motion for an unsteady one-dimensional open channel flow with free surface $h(x,t)$ are, in the usual notation:

$$\frac{\partial h}{\partial t}+\frac{\partial}{\partial x}(hu)=0,\quad \frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=-g\frac{\partial h}{\partial x}.$$

(i) Show that these equations may be written in the alternative form

$$\begin{aligned}\left(\frac{\partial}{\partial t}+(u+c)\frac{\partial}{\partial x}\right)(u+2c) &= 0,\\ \left(\frac{\partial}{\partial t}+(u-c)\frac{\partial}{\partial x}\right)(u-2c) &= 0,\end{aligned}$$

where $c^2=gh$.

(ii) Suppose that $u=u_0$ when $c=c_0$. Deduce that these equations admit a solution with $(u-2c)$ constant everywhere, and u constant along the characteristics

$$\frac{dx}{dt}=\frac{3}{2}u-\frac{1}{2}u_0+c_0.$$

(b) A semi-infinite region of water of uniform depth h_0 extends to infinity in the positive $x-$direction, is at rest initially, and is bounded at the end $x=0$ by a vertical wall. At time $t=0$ the wall starts to move in the negative $x-$direction with speed $V_0\tanh(\alpha t)$, where V_0 and α are constants.

(i) Calculate the position $X(t)$ of the wall at time t.

(ii) Assuming a solution of the form given in (a)(ii), show that at time t the depth of the water is given by

$$h(x,t)=h_0\left(1-\frac{V_0}{2c_0}\tanh(\alpha\tau)\right)^2,\quad (X(t)\le x\le c_0t),$$

where $c_0^2=gh_0$ and τ is given in terms of x and t by the relation

$$x+\frac{V_0}{\alpha}\ln\left(\cosh(\alpha\tau)\right)=\left(c_0-\frac{3}{2}V_0\tanh(\alpha\tau)\right)(t-\tau).$$