1. (a) Derive the Euler-Lagrange equations satisfied by extrema of the functional of a scalar variable $y(t)$:

$$F[y] = \int_a^b f(y, y_t, t)\mathrm{d}t.$$

Explain your argument clearly, and in particular explain what end condition you impose on $y(a)$ and $y(b)$. How is the Euler-Lagrange equation generalised to more than one dependent variable?

(b) A uniform light rod of length ℓ moves in the plane, without any external forces. A mass M is attached to each end, at $\mathbf{x}_1$ and $\mathbf{x}_2$. How many degrees of freedom does the system have?

Explain why the Lagrangian may be written as:

$$L = \frac{M}{2}(|\dot{\mathbf{x}}_1|^2 + |\dot{\mathbf{x}}_2|^2) - \frac{\lambda(t)}{2}(|\mathbf{x}_1 - \mathbf{x}_2|^2 - \ell^2).$$

Explain the physical meaning of $\lambda(t)$. Write down the action integral, and hence find the set of Euler-Lagrange equations for this system.

Using the constraint and its derivatives, find an explicit expression for $\lambda(t)$.

2. (a) Explain, briefly, what is meant by a *symmetry* of a system with Lagrangian $L(\mathbf{y}, \mathbf{y}_t, t)$.

(b) A mass m is free to move along the x-axis, having coordinates $(x, 0)$. It is joined by a light extensible string with constant tension T to a second mass m which is free to move in the plane, having coordinates (X, Y). Gravity g acts in the negative y-direction.

Write down the Lagrangian for the system. Identify the symmetries of the system, and construct a conserved quantity corresponding to each of them.

Discuss whether the system is integrable.

3. (a) State and prove Euler's theorem for homogeneous functions. Explain how it may be used to calculate the energy of a system with Lagrangian

$$L = L_2 + L_1 + L_0,$$

where each of the L_i is homogeneous of degree i in the velocities $\dot{\mathbf{q}}$.

Explain what is meant by the *Hamiltonian* of a Lagrangian system with Lagrangian L.

(b) A particle of mass m and charge e moves in the plane with Lagrangian

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) + eA_x\dot{x} + eA_y\dot{y},$$

where A_x and A_y are specified functions of (x, y). Find the Hamiltonian of the system. If

$$A_x = y, \qquad A_y = -x,$$

where $r = \sqrt{x^2 + y^2}$, show that the system is symmetric under rotations about the origin, and hence find a second conserved quantity of the system.

Verify that this quantity and the Hamiltonian have zero Poisson bracket.

4. (a) Explain what is meant by a *normal mode* of an equilibrium of a Lagrangian system.

(b) *'The Acetylene molecule'*

Four masses m, M, M and m in a line are coupled by springs with spring constants k, K and k. The unstretched length of each spring is a.

$$\begin{array}{ccccccc} x_1 & & x_2 & & x_3 & & x_4 \\ \rightarrow & & \rightarrow & & \rightarrow & & \rightarrow \\ \bullet & & \bullet & & \bullet & & \bullet \\ m & k & M & K & M & k & m \end{array} \tag{1}$$

If the four masses are given small displacements from equilibrium, along the line, of magnitude x_1, x_2, x_3 and x_4 respectively, write down a Lagrangian for the system.

Explain why you would expect the normal modes of the system to have either the form

$$\begin{pmatrix} x \\ X \\ X \\ x \end{pmatrix},$$

or else

$$\begin{pmatrix} x \\ X \\ -X \\ -x \end{pmatrix}.$$

Hence calculate the frequencies of oscillation for each class of mode, obtaining the characteristic equation for each class separately. Find all the modes and sketch the corresponding motions of the system.

5. A point mass m with instantaneous position vector $\mathbf{x}(t)$ rotates about the origin with angular velocity vector ω. Write down the kinetic energy of the particle in the form

$$T = \frac{1}{2}\omega^T I \omega,$$

giving the 3×3 matrix I, the inertia tensor, explicitly. Explain how this formula extends to rigid collections of point masses. Further, express the angular momentum of the mass about the origin in terms of ω.

Calculate the inertia tensor for a uniform rectangular plane body, of mass M, with vertices $(\pm a, \pm b, 0)$, with $a > b$.

Write down Euler's equations describing the free rotation of this body, explaining the meaning of the terms.

The body rotates steadily about a fixed axis, without an applied couple. For which axes is this possible? Discuss whether, and under what conditions, this motion is stable.