Imperial College London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2006

This paper is also taken for the relevant examination for the Associateship.

M2A2

Dynamics I

Date: Tuesday, 23rd May 2006 Time: 2 pm – 4 pm

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. A function $y(x)$ has the property that the functional

$$F[y] = \int_a^b f(y, y_x, x)\mathrm{d}x$$

is stationary with respect to all variations

$$y(x) \to y(x) + \epsilon\eta(x)$$

which keep $y(a)$ and $y(b)$ fixed. Show how this property leads to the Euler-Lagrange equation satisfied by $y(x)$.

A mass m moves in the plane along the curve given in polar coordinates by

$$r = f(\theta).$$

It is released from rest at the point $(r, \theta) = (f(0), 0)$, and moves with constant total energy under the potential

$$V = mgr.$$

Write down as an integral $\int_0^{\theta_1} L(f, f', \theta)\mathrm{d}\theta$, the time taken for the particle to reach the point $(r, \theta) = (f(\theta_1), \theta_1)$, and write down the Euler-Lagrange equation which f must satisfy if this time is minimised.

Identify a symmetry of the problem, and hence, stating any general result that you use, integrate the Euler-Lagrange equation once, and reduce it to a first order differential equation. *Do not attempt to solve this ordinary differential equation.*

2. Explain what is meant by a *symmetry* of a Lagrangian

$$L(\mathbf{q}, \dot{\mathbf{q}}, t),$$

and state Noether's theorem.

A simplified model of a water molecule consists of a mass M at the point $\mathbf{x}_0$ in the plane, and two other masses m at points $\mathbf{x}_1$ and $\mathbf{x}_2$. These two are joined to the mass M by rigid rods of length a.

The only potential in the system is a function $V(\theta)$, where θ is the angle between the two rods.

Write down a Lagrangian for this constrained system. How many degrees of freedom does the system have?

Identify six symmetries of the system, and write down the corresponding integrals of the motion. In particular find the conserved quantity corresponding to the Galilean invariance of L. Show how the number of degrees of freedom may thus be reduced by going to a moving frame of reference, and eliminating $\mathbf{x}_0$.

3. (a) Explain briefly how, given the Euler-Lagrange equations corresponding to a Lagrangian

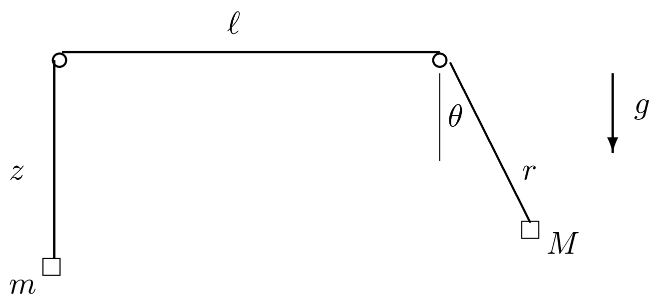
$$L(\mathbf{q}, \dot{\mathbf{q}}, t),$$

one may define a function

$$H(\mathbf{q}, \mathbf{p}, t),$$

and a set of equations of motion for $\mathbf{q}$ and $\mathbf{p}$ which are equivalent to the Euler-Lagrange equations. State clearly how the function $H(\mathbf{q}, \mathbf{p}, t)$ and the new variables $\mathbf{p}$ are defined.

(b) Two masses are suspended from an inextensible string of length $\ell + a$. This passes over two small smooth pulleys a distance ℓ apart, and from one end, a mass m is constrained to hang vertically downwards from the pulley, while from the other, a mass M hangs, which is free to move in the plane. Gravity acts vertically downwards.



Using coordinates as in the diagram, write down a Lagrangian for the system, using the constraint to eliminate z and $\dot{z}$.

Find the Hamiltonian corresponding to this Lagrangian, and write down Hamilton's equations.

4. Define an *equilibrium* of a Lagrangian dynamical system, and derive the conditions which any equilibrium configuration must satisfy.

Three pendula are coupled by a nearest neighbour potential. The Lagrangian of the system is:

$$L = \frac{I}{2}\left(\dot{\theta}_1^2 + \dot{\theta}_2^2 + \dot{\theta}_3^2\right) + k\left(\cos(\theta_1) + \cos(\theta_1 - \theta_2) + \cos(\theta_2) + \cos(\theta_2 - \theta_3) + \cos(\theta_3)\right),$$

where I and k are constants.

Find the equilibrium conditions for this system.

Verify that the configuration $(\theta_1, \theta_2, \theta_3) = (0, 0, 0)$ is an equilibrium, and calculate the Lagrangian $L^{(2)}$ describing small perturbations about this equilibrium.

Hence find the normal modes of the system about this configuration. Is the linearised motion resulting from a general small perturbation of this state (a) bounded, (b) periodic?

5. A collection of masses m_i at points $\mathbf{r}_i(t)$ rotates rigidly about a fixed point $\mathbf{r}_0$, with angular velocity (in space) $\boldsymbol{\omega}$. Calculate the angular momentum of the system about the point $\mathbf{r}_0$ in terms of $\boldsymbol{\omega}$. Hence define the inertia tensor of the system and write down an explicit expression for it.

Calculate the inertia tensor, about the origin, of a system consisting of four masses m at the two points $(\pm a, 0, 0)$, and the two points $(0, \pm b, 0)$.

If $b > a$, find the principal axes of the inertia tensor, and the corresponding principal moments of inertia. State about which axes you would expect the body to rotate freely without an applied couple. Would such a motion always be stable? If not, state about which axis or axes you would expect the rotation to be unstable.