1. (a) Show that if in a variational problem, to extremise

$$\int_{x_0}^{x_1} F(y, y_x, x)\mathrm{d}x,$$

it is given that

$$\frac{\partial F}{\partial x} \equiv 0,$$

then the 'energy' E:

$$E = y_x \frac{\partial F}{\partial y_x} - F$$

is constant if y satisfies the Euler-Lagrange equation.

(b) A surface of revolution $\mathcal{S}$ is expressed in cylindrical polar coordinates (r, θ, z) as

$$r = f(z), \qquad -b \leq z \leq b.$$

Such a surface intersects the planes $z = \pm b$ in circles $r = a$. You may assume that $f(z) > 0$ for all $z \in [-b, b]$. Write down, as an integral with respect to z, the area A of the surface $\mathcal{S}$.

If the surface is such as to minimise A, write down the Euler-Lagrange equation which $f(z)$ must satisfy. Show that this Euler-Lagrange equation has a first integral,

$$g(f, f_z) = \text{constant}.$$

Find this integral, and hence show that the general solution of this Euler-Lagrange equation is

$$f(z) = \frac{1}{k}\cosh(kz - l),$$

where k and l are arbitrary constants. Find the solution satisfying the given boundary conditions, $r = a$ at $z = \pm b$. Show, graphically or otherwise, that there is no solution to the problem if b/a is too large.

2. N particles of masses m_i, and position vectors $\mathbf{x}_i$, $\quad i = 1, \ldots, N$, move in the plane, interacting pairwise via potentials $V_{ij}(|\mathbf{x}_i - \mathbf{x}_j|)$. There is no external potential.

   (a) Write down the Lagrangian of the system.

   (b) Describe the symmetries of the system, explaining carefully what you mean by a symmetry. Hence construct 5 integrals of motion for the system.

   (c) Show that after a simple change of coordinates, three of these integrals may be fixed to be zero.

   (d) In the planar 3-body problem, $N = 3$, and $V_{ij} = -\dfrac{Gm_i m_j}{|\mathbf{x}_i - \mathbf{x}_j|}$.

   Given that no other integrals of motion are known in this case, discuss briefly whether or not the system is integrable.

3. A rod is inclined at an angle α to the horizontal. A mass m is free to slide along it. A mass m hangs from the first mass, on a string of length l. Gravity acts vertically downwards.

   (a) Show that the Lagrangian L for the system, using coordinates as in the diagram, is given by:

$$L = m\dot{X}^2 + ml\dot{X}\dot{\theta}\cos(\theta - \alpha) + \frac{m}{2}l^2\dot{\theta}^2 + mgl\cos(\theta) - 2mgX\sin(\alpha).$$

   (b) Derive the two Euler-Lagrange equations for the system.

   (c) Calculate the energy E of the system, and hence write down the Hamiltonian.

   (d) Show that translation in X is a symmetry of L; hence or otherwise construct a second integral of motion.

4. A light wheel of radius a has a mass M fixed to the lowest point of its circumference. The wheel is free to rotate about its centre. A string passes over the wheel; one end is attached to a mass M, while on the other side the end is attached to one end of a spring with spring constant Mk. A third mass M hangs at the spring's lower end.

(a) Explain the derivation of the Lagrangian

$$L = \frac{M}{2}(2\dot{Z}_1^2 + \dot{Z}_2^2) - \frac{kM}{2}(Z_1 + Z_2)^2 + Mga\cos(\frac{Z_1}{a})$$

for the system; you should assume that the string does not slip against the wheel, and that the hanging masses move only vertically. Take as coordinates the downward vertical displacements of the two hanging masses from their equilibrium positions.

(b) Find the quadratic approximation $L^{(2)}$ to the Lagrangian near the equilibrium, and hence find the frequencies of small oscillations of the system.

5. A point mass m with position vector $\mathbf{x}$ rotates about the origin with angular velocity vector $\boldsymbol{\omega}$.

(a) Write down the kinetic energy of the particle in the form

$$\boldsymbol{T = \frac{1}{2}\omega^T I \omega,}$$

giving the 3×3 matrix I, the inertia tensor, explicitly. Hence write down the inertia tensor for a system of N rigidly connected point masses, m_i at points $\mathbf{x}_i$, satisfying $|\mathbf{x}_i - \mathbf{x}_j| =$ constant.

What is the angular momentum of the system if the angular velocity is $\boldsymbol{\omega}$?

(b) State Euler's equations for a rotating body. What relation do the inertia tensor, angular velocity and angular momentum defined in part (a) of the question have with the analogous quantities appearing in Euler's equations?

(c) A body consists of 2 masses M and 2 masses m, with $M > m$, rigidly fixed to a light square frame as shown in the diagram

Write down its inertia tensor in the (X_1, X_2, X_3) coordinates. Write down Euler's equations for this system, state what steady rotations are possible, and state, with reasons, which ones are stable.