UNIVERSITY OF LONDON

Course:	M2A2
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BSc and MSci EXAMINATIONS (MATHEMATICS)
MAY–JUNE 2004

This paper is also taken for the relevant examination for the Associateship.

M2A2 Dynamics I

Date: examdate   Time: examtime

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

Statistical tables will not be available.

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1. A function $y(x)$ has the property that the functional

$$F[y] = \int_a^b f(y, y_x, x)\mathrm{d})x$$

is stationary with respect to all variations

$$y(x) \to y(x) + \epsilon\eta(x)$$

which keep $y(a)$ and $y(b)$ fixed. Show how this property leads to the Euler-Lagrange equation satisfied by $y(x)$.

In the case that f is independent of x, show that the Euler-Lagrange equation has an *integral* - a function $g(y, y_x)$ such that

$$\frac{\mathrm{d}g}{\mathrm{d}x} = 0.$$

What is the relationship between $f(y, y_x)$ and $g(y, y_x)$?

A surface of revolution $\mathcal{S}$ is given by

$$r = \cosh(z)$$

in cylindrical polar coordinates (r, z, ϕ). A curve, lying in $\mathcal{S}$, is given by:

$$z = z(\phi)$$

and is chosen to connect the two points (z_1, ϕ_1), (z_2, ϕ_2). Write down the total length of this curve as an integral in the form

$$L = \int_{\phi_1}^{\phi_2} h(z, \frac{\mathrm{d}z}{\mathrm{d}\phi}, \phi)\mathrm{d}\phi.$$

Find the Euler-Lagrange equation satisfied by extrema of L. Reduce this equation to one of first order, and hence show that the extrema may be given as:

$$\int_{z_1}^{z} \frac{\mathrm{d}z}{\sqrt{K^2\cosh^2(z) - 1}} = \int_{\phi_1}^{\phi} \mathrm{d}\phi,$$

where K is some constant.

2. Two particles with masses m_1, m_2 and position vectors $\mathbf{x}_1$, $\mathbf{x}_2$, move in $\mathbb{R}^3$, and interact via a potential $V(|\mathbf{x}_1 - \mathbf{x}_2|)$. Write down a Lagrangian for this system and find the corresponding Hamiltonian. By changing coordinates to:

$$\mathbf{X} = \frac{1}{m_1 + m_2}(m_1\mathbf{x}_1 + m_2\mathbf{x}_2)$$

$$\mathbf{r} = \mathbf{x}_1 - \mathbf{x}_2,$$

show that the Lagrangian decouples into two independent parts, each depending on either $\mathbf{X}$ and its derivatives or $\mathbf{r}$ and its derivatives. Show that the quantity

$$\mathbf{k} = \mathbf{r} \wedge \dot{\mathbf{r}}$$

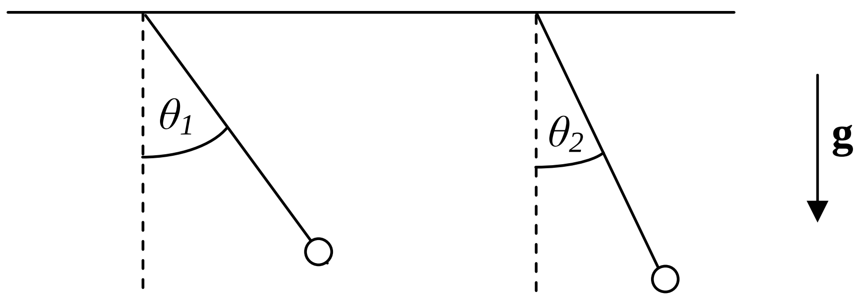
is conserved, and hence show that $\mathbf{r}(t)$ is confined to a certain plane for all t.

By transforming to polar coordinates in this plane, show that the radial and angular motions also decouple. Show that the solution of the radial motion is:

$$\int_{r_0}^{r} \frac{\mathrm{d}r}{\sqrt{2(E - V(r))\frac{m_1+m_2}{m_1 m_2} - \frac{|\mathbf{k}|^2}{r^2}}} = \int_0^t \mathrm{d}t,$$

where E is an arbitrary constant.

3. Two identical pendula of mass m and length l swing from a beam.



Their motions are coupled by an additional potential $k\theta_1\theta_2$. Write down a Lagrangian for the combined system, and find the Euler-Lagrange equations.

Find the equations satisfied by equilibrium points of the system and verify that $\theta_1 = \theta_2 = 0$ is an equilibrium. Expand the Lagrangian for small disturbances from this equilibrium, and find the normal modes and their frequencies. What is the condition for this equilibrium to be stable?

4. Euler's equations for a rigid body rotating about the origin are:

$$\frac{\mathrm{d}\mathbf{J}}{\mathrm{dt}} = -\mathbf{\Omega} \wedge \mathbf{J} + \mathbf{G}.$$

Explain the meaning of each term in the equation, and sketch the key steps in its derivation. How would $\mathbf{\Omega}$and $\boldsymbol{J}$be related?

A rigid body is symmetrical about its third axis. It rotates freely about its centre of mass, so its Lagrangian in terms of the Euler angles is:

$$L = \frac{1}{2} I_1(\dot{\theta}^2 + \sin^2(\theta)\dot{\phi}^2) + \frac{1}{2} I_3(\dot{\psi} + \cos(\theta)\dot{\phi})^2.$$

Identify the symmetries of this Lagrangian and find the corresponding conserved quantities.

Find the Hamiltonian corresponding to L, and hence show that if $y = \cos(\theta)$, then it satisfies:

$$\dot{y}^2 = \alpha + \beta y + \gamma y^2,$$

where α, β, γ are constants of integration.

5. A heavy inextensible rope of constant mass density ρ per unit length and total length l, moves in the (x, y) plane. One end is attached to a fixed point at the origin, while the other end is free to move. The curve of the rope at time t is given parametrically by the functions $(x(s,t), y(s,t))$. Take s to be the arclength along the rope, measured from the fixed end.

Write down the kinetic energy of the rope, and the constraint which the functions $x(s,t)$ and $y(s,t)$ must satisfy, and hence show that, if no external forces act on the rope, a Lagrangian for the constrained system is:

$$L = \int_0^l \frac{\rho}{2}\left(\frac{\partial x}{\partial t}^2 + \frac{\partial y}{\partial t}^2\right) - \frac{\lambda(s,t)}{2}\left(\frac{\partial x}{\partial s}^2 + \frac{\partial y}{\partial s}^2\right) \mathrm{d}s.$$

The equations of motion are found by looking for extrema of

$$\int_{t_1}^{t_2} L \mathrm{d}t$$

with respect to variations

$$x(s,t) \to x(s,t) + \epsilon\xi(s,t),$$

$$y(s,t) \to y(s,t) + \epsilon\eta(s,t),$$

and requiring (ξ, η) to vanish at the boundaries $t = t_1$, $t = t_2$, and at the fixed end $s = 0$. Obtain the Euler-Lagrange equations of the system from first principles; by considering the boundary term at the free end $s = l$, show that the Lagrange multiplier λ must vanish at this point.

By eliminating the Lagrange multiplier using the constraint, show that

$$\frac{\partial^2 \mathbf{x}}{\partial t^2} = \frac{\partial}{\partial s}\left(\frac{\partial \mathbf{x}}{\partial s} \int_l^s \frac{\partial^2 \mathbf{x}}{\partial t^2} \cdot \frac{\partial \mathbf{x}}{\partial s'} \mathrm{d}s'\right).$$