M2A1 Exam 2007

1. Consider the pair of ordinary differential equations

$$\dot{x} = -x - xy\,, \qquad\qquad \dot{y} = 1 - x - y\,,$$

where the dot refers to differentiation with respect to time t.

(i) Show that they have two fixed points, one of which lies on the y-axis and the other lies in the lower half-plane.

(ii) Find the local eigenvalues and eigenvectors of the Jacobian matrices corresponding to each point. Hence show that these points are classified as a stable node and a saddle respectively.

(iii) Sketch the phase plane showing how phase trajectories connect between these two points.

(iv) In particular sketch the trajectory in the upper half-plane that is is tangent to the x-axis at the point $(1,\,0)$. Use the differential equations to show that near this point, this trajectory can be approximated by

$$y = \tfrac{1}{2}(x-1)^2\,.$$

2. Find the critical points of the nonlinear system of ordinary differential equations

$$\dot{x} = y\,, \qquad\qquad \dot{y} = y + 1 - x^2\,.$$

Classify these critical points and sketch a selection of orbits in the phase plane. On your diagram, you should show the locus of points on which $dy/dx = 0$, the locus of points on which $dy/dx = \infty$, and the signs of dy/dx in each sector into which the phase plane is divided by these loci.

3. Use the disturbance function method to show that the integral

$$I = \int_{x_1}^{x_2} f(x, y, y')\,dx$$

takes stationary values if $y(x)$ satisfies the Euler-Lagrange equation

$$f_y - \frac{d}{dx} f_{y'} = 0\,.$$

$y(x_1)$ and $y(x_2)$ take fixed values and subscripts denote partial derivatives. You may assume that f has continuous 2nd partial derivatives.

(i) Write down an expression for the total derivative d/dx.

(ii) Show that the Euler-Lagrange equation can be written in the form

$$f_x + \frac{d}{dx}\left(y' f_{y'} - f\right) = 0.$$

Hence show that if $y(0) = 0$ and $y(1) = 1$ and

$$I = \int_0^1 \left(y'^2 + \beta^2 y^2\right)\, dx\,,$$

then I takes stationary values when $y(x)$ satisfies

$$y = \frac{\sinh \beta x}{\sinh \beta}.$$

4. The positive part of the y-axis is the principal axis of a cone whose vertex lies at the origin. The curved surface of the cone is represented by

$$x^2 + z^2 = y^2\,,$$

which is parametrized by $x = y\cos\theta$ and $z = y\sin\theta$. Show that the arc length of a curve on this surface, with end points represented by θ_1 and θ_2, is given by

$$\text{arc length} = \int_{\theta_1}^{\theta_2} ds = \int_{\theta_1}^{\theta_2} \left\{2\left(\frac{dy}{d\theta}\right)^2 + y^2\right\}^{1/2} d\theta\,.$$

Show that the arc length takes stationary values when y and θ satisfy the differential equation

$$2c^2\left(\frac{dy}{d\theta}\right)^2 = y^2\left(y^2 - c^2\right)\,, \qquad c = \text{const}\,.$$

Show that this differential equation is satisfied by

$$y = \pm c\,\text{cosec}\left(\frac{\theta + \delta}{\sqrt{2}}\right)\,, \qquad \delta = \text{const}\,.$$

You may assume the Euler-Lagrange equations in the form $(y' = dy/d\theta)$

$$f_\theta + \frac{d}{d\theta}\left(y' f_{y'} - f\right) = 0.$$

5. Traffic flows on a road with a density $\rho(x,t)$ according to the kinematic wave equation

$$\rho_t + (1-\rho)\rho_x = 0\,.$$

(a) Show that for initial data $\rho(x,0) = f(x)$ there is an implicit solution of the form

$$\rho = f(x - (1-\rho)t)\,.$$

(b) Consider the two sets of initial conditions:

(i)

$$f(x) = \begin{cases} x & 0 \le x \le 1\,, \\ 2 - x & 1 \le x \le 2\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

(ii)

$$f(x) = \begin{cases} 1 & x \le 0\,, \\ \frac{1}{1+x} & x \ge 0\,. \end{cases}$$

Show that a shock develops from (i) but not from (ii). Does this shock develop on the backwards-looking face or the forwards-looking face?

(c) For case (i) in Part (b) above, find the solution $\rho(x,t)$ explicitly in terms of x and t.

(d) Show that for this version of the kinematic wave equation, the propagation velocity is less than the flow velocity.