Imperial College London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2006

This paper is also taken for the relevant examination for the Associateship.

M2A1

Applied Mathematics

Date: Tuesday, 16th May 2006 Time: 10 am – 12 noon

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. Consider the following pair of non-linear ordinary differential equations

$$\dot{x} = x - xy\,, \qquad\qquad \dot{y} = -y + x^2 y\,.$$

(i) Find the critical points and determine their nature.

(ii) Draw in the lines $x = \pm 1$ and $y = 1$ in the (x, y)-plane. Together with the axes, these lines divide the phase plane into 12 regions. Mark in each of these 12 regions the sign of dy/dx, together with the locus of points on which $dy/dx = 0$ and the locus of points on which $dy/dx = \infty$.

(iii) Sketch the phase trajectories.

2. Consider the variational problem

$$I = \int_{x_1}^{x_2} f\left(x,\, y(x),\, y'(x),\, y''(x)\right)\, dx$$

where $y(x)$ and $y'(x)$ are fixed at the end points x_1 and x_2 and $y(x)$ has three continuous derivatives.

(i) Use the disturbance function method to show that if $y(x)$ minimizes I then the Euler-Lagrange equation for this problem is

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right) = 0\,.$$

In deriving this, certain restrictions must be placed upon the disturbance function $\eta(x)$. What are these?

(ii) Write out explicitly the operator d/dx in terms of the partial derivatives of x, y, y' and y''.

3. A curve $y = y(x)$ lies in the first quadrant of the (x, y)-plane and has end-points at $(0, 0)$ and $(a, 0)$. By rotation about the x-axis this curve forms a solid of revolution whose volume V and surface area A are respectively $(y' = dy/dx)$

$$V = \pi \int_0^a y^2 dx\,, \qquad\qquad A = 2\pi \int_0^a y\,(1 + y'^2)^{1/2} dx\,.$$

Use the method of constrained multipliers to show that the sphere is the solid figure of revolution which, for a fixed surface area, has maximum volume.

Hint: use the end conditions $(0, 0)$ and $(a, 0)$ to show that the constant of integration appearing in the integrated constrained Euler-Lagrange equation is zero.

4. The wave equation

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} = 0$$

on the infinite line has initial shape and velocity profiles given by

$$y(x,\,0) = F(x) \qquad \text{and} \qquad \frac{\partial y(x,0)}{\partial t} = G(x)\,.$$

(i) Derive d'Alembert's general solution

$$y(x,t) = \tfrac{1}{2}\left[F(x-ct) + F(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} G(s)\,ds\,.$$

(ii) If initial conditions are such that

$$F(x) = \sum_{n=0}^{\infty} a_n \cos(k_n x) \qquad \text{and} \qquad G(x) = c\sum_{n=0}^{\infty} k_n a_n \cos(k_n x)\,,$$

show that for $t > 0$

$$y(x,t) = \sqrt{2}\sum_{n=0}^{\infty} a_n \cos(k_n x)\sin\left(k_n ct + \frac{\pi}{4}\right)\,.$$

5. A nonlinear oscillator has equation of motion

$$\ddot{x} = F(x)\,,$$

where $F(x)$ has at least one continuous derivative.

(i) What is the condition on $F(x)$ for critical points to be saddles? If $y = \dot{x}$, show that the equations for the separatrices are given by

$$y^2 = 2\int_{x_0^{(s)}}^{x} F(s)\,ds\,,$$

where $x_0^{(s)}$ in the lower limit represents that class of critical points that are saddles.

(ii) If $F(x)$ is given by

$$F(x) = \sin x - \sin 2x\,,$$

show that the equations for the separatrices are

$$\sqrt{2}y = \pm(2\cos x - 1)\,.$$