M21 Exam: May 2005

1. (a) L. F. Richardson's model of how the armaments $x(t)$ and $y(t)$ produced by two opposing nations evolve according to time is represented by the pair of ordinary differential equations

$$\dot{x} = ay - x + g_1 \qquad \dot{y} = ax - y + g_2 \qquad a > 0\,.$$

The pair of terms ay and ax represent the fear induced by the two respective opponents, the terms $-x$, $-y$ represent the deterrent cost of the armaments, and $g_1 > 0$ and $g_2 > 0$ represent the constant grievances that each hold against the other.

(i) Show that the single fixed point takes the form of a stable node in the first quadrant when $0 < a < 1$ and a saddle in the third quadrant when $a > 1$.

(ii) Find the directions of the eigenvectors in these cases.

(b) Richardson's model is now modified such that not only are grievances removed but the deterrent cost of the armaments is no longer linear but quadratic, giving equations

$$\dot{x} = ay - x^2 \qquad \dot{y} = ax - y^2\,.$$

(iii) Show that for all values of $a > 0$ $(a \neq 1)$ there are now two fixed points, one a saddle and the other a stable node.

(iv) Sketch the phase portrait together with the direction of the eigenvectors and the signs of dy/dx in different sectors of the phase plane.

(v) What fundamental difference in behaviour has the change in the model created?

2. Throughout the question f is a function of $2n + 1$ variables : t, $q_1(t)$, $q_2(t)$, $\ldots q_n(t)$ and $\dot{q}_1(t)$, $\dot{q}_2(t)$, $\ldots \dot{q}_n(t)$ and is written as $f(t,\, q_i,\, \dot{q}_i)$ in shorthand notation. It has first partial derivatives in all $2n + 1$ variables. The dot refers to differentiation with respect to t.

(a) Show that if the quantities $q_i(t)$ minimize the integral

$$I = \int_{t_1}^{t_2} f(t,\, q_i,\, \dot{q}_i)\, dt$$

for fixed end-points t_1 and t_2, then they satisfy n separate Euler-Lagrange equations

$$\frac{\partial f}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{q}_i}\right) = 0 \qquad i = 1,\, 2,\, \ldots,\, n\,.$$

(b) Using the chain rule, show that the total derivative d/dt is

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \sum_{i=1}^{n}\left(\dot{q}_i \frac{\partial}{\partial q_i} + \ddot{q}_i \frac{\partial}{\partial \dot{q}_i}\right),$$

and hence show that

$$\sum_{i=1}^{n} \frac{d}{dt}\left(\dot{q}_i \frac{\partial f}{\partial \dot{q}_i} - f\right) + \frac{\partial f}{\partial t} = 0\,.$$

3. An SIR disease model relating the number of susceptibles $x(t)$, infectives $y(t)$ and recovered $z(t)$, in a population of magnitude N, is governed by the set of ordinary differential equations

$$\dot{x} = -x^2y + a \qquad \dot{y} = x^2y - b^2y \qquad \dot{z} = F(x,y)$$

subject to a constant total population $x + y + z = N$ and $a,\ b > 0$.

(a) Show that there is only one critical point in the first quadrant of the $x - y$ phase plane and that this is a stable node when $a > 2b^3$ and a stable spiral when $a < 2b^3$.

(b) When $a < 2b^3$, sketch some typical orbits in the first quadrant of the phase plane, marking the sign of dy/dx in appropriate regions.

(c) What is the value of z when $t \to \infty$?

(d) What is $F(x,y)$?

4. The curved surface of a parabolic cone has the equation $z = \frac{1}{2}(x^2 + y^2)$.

(a) Use polar co-ordinates $x = r\cos\theta;\quad y = r\sin\theta$ to show that a small element of arc length ds on the surface of the cone can be expressed as

$$(ds)^2 = (1 + r^2)(dr)^2 + r^2(d\theta)^2\,.$$

(b) Show that the problem of finding the geodesics on the surface of the cone can be expressed as finding those curves $r = r(\theta)$ that minimize the integral

$$\int_{\theta_1}^{\theta_2} f(r, r')\,d\theta$$

where $f(r, r')$ is to be found $(r' = dr/d\theta)$.

(c) Use the Euler-Lagrange equation to show that geodesics must satisfy

$$c\int \frac{1}{r}\sqrt{\frac{r^2+1}{r^2-c^2}}\,dr = \theta + \delta\,, \qquad c \text{ and } \delta \text{ arbitrary constants.}$$

With $f = f(\theta, r, r')$, the Euler-Lagrange equation is $f_r - \frac{d}{d\theta}f_{r'} = 0$ or, in its alternative form, $f_\theta + \frac{d}{d\theta}(r'f_{r'} - f) = 0$. Subscripts are partial derivatives.

5. The kinematic wave equation for the density function $\rho(x,t)$ is

$$\rho_t + c(\rho)\rho_x = 0\,,$$

where $c(\rho)$ is a smooth function and the subscripts x and t are partial derivatives.

(a) Show that with initial data $\rho(x,0) = f(x)$ for some given differentiable function f, the implicit solution $\rho = \rho(x,t)$ of the kinematic wave equation can be written in the form

$$\rho = f\left(x - c(\rho)t\right)\,.$$

(b) Initial data $f(x)$ is defined in four different regions by

$$f(x) = \begin{cases} 2-x & 1 \le x \le 2 \\ 1 & -1 \le x \le 1 \\ 2+x & -2 \le x \le -1 \\ 0 & \text{otherwise} \end{cases}$$

and $c(\rho) = 1 - \rho$. This has four apexes ; at $x = -2,\ -1,\ 1$ and 2.

(i) Find $\rho(x,t)$ as an explicit function of x and t in all four regions.

(ii) At what time $t^* > 0$ does $\rho_x = \infty$ and on what face does this occur?

(iii) Sketch ρ versus x for the four times : $t = 0,\ \ 0 < t < t^*,\ \ t = t^*$ and $t > t^*$. Find the relation $x = x(t)$ for how each of the four apexes move.