M2A1 – 2004

[**1**] For the following system of non-linear ordinary differential equations:

$$\dot{x} = x - xy \qquad\qquad \dot{y} = -y + x^2 y,$$

find the critical points and determine their nature. Mark in each apropriate region of the phase plane the sign of dy/dx, together with the locus of points on which $dy/dx = 0$ and the locus of points on which $dy/dx = \infty$.

Sketch the phase plane drawing a few trajectories in each quadrant.

Show that trajectories represent solutions of the transcendental equation

$$\ln|xy| = y + \tfrac{1}{2}x^2 + c$$

where c is a constant which is dependent on initial conditions.

[**2**] Show that the equations for the separatrices of the nonlinear oscillator

$$\ddot{x} = F(x)$$

are given by $(y = \dot{x})$

$$\tfrac{1}{2}y^2 = \int_{x_0^{(s)}}^{x} F(x')\, dx'$$

where $x_0^{(s)}$ in the lower limit represents a certain class of critical points. What class of critical points do these need to be and what is the corresponding condition on $F(x)$?

If $F(x)$ is given by

$$F(x) = \sin x - \sin 2x$$

show that the equations for the separatrices are

$$\sqrt{2}y = \pm(2\cos x - 1)\,.$$

Hence sketch the phase plane, including these separatrices.

[**3**] A guitar string of length L and constant density ρ is clamped at both ends with a tension T. It undergoes small transverse oscillations $y(x,t)$ which are governed by the wave equation

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} = 0\,,$$

where $c^2 = T/\rho$. Given the end conditions, use the method of separation of variables to show that the general solution is

$$y(x,t) = \sum_{n=1}^{\infty} R_n \sin\left(\frac{n\pi x}{L}\right)\sin\left(\omega_n t + \delta_n\right)$$

where $\omega_n = n\pi c/L$ and R_n and δ_n are arbitrary constants.

Given that the total energy in the string is given by

$$E = \tfrac{1}{2}T\int_0^L \left\{\left(\frac{\partial y}{\partial x}\right)^2 + \frac{1}{c^2}\left(\frac{\partial y}{\partial t}\right)^2\right\} dx$$

use the above solution to show that

$$E = \frac{\pi^2 T}{4L}\sum_{n=1}^{\infty} n^2 R_n^2\,.$$

Orthogonality relations between the sines and cosines for integers n and m are given by

$$\int_0^L \begin{matrix}\sin\\ \cos\end{matrix}\left(\frac{n\pi x}{L}\right)\begin{matrix}\sin\\ \cos\end{matrix}\left(\frac{m\pi x}{L}\right) dx = \tfrac{1}{2}L\delta_{nm}\,.$$

[**4**] (i) The kinematic wave equation for the density of a quantity $\rho(x,t)$ is governed by

$$\rho_t + c(\rho)\rho_x = 0$$

where $c(\rho)$ is the propagation velocity. Initial data is specified by $\rho(x,0) = f(x)$ where $f(x)$ is some continuous function in $-\infty \le x \le \infty$. Show that ρ_x and ρ_t can only become simultaneously infinite at some positive time t^* if $c'f' < 0$ somewhere in $-\infty \le x \le \infty$.

(ii) Using the notation $y' = dy/dx$, consider the functional

$$I[y(x)] = \int_a^b f(x,y,y')\,dx,$$

where $f(x,y,y')$ is at most quadratic in y and y' and has continuous second derivatives. The values of $y(a)$ and $y(b)$ are fixed. Let the stationary function of I be $Y(x)$ and let $h(x)$ be any function with a continuous first derivative satisfying $h(a) = h(b) = 0$. Use a double Taylor expansion to show that

$$I[Y(x) + h(x)] - I[Y(x)] = \tfrac{1}{2}\int_a^b \left[f_{yy}h^2 + 2hh'f_{yy'} + f_{y'y'}h'^2\right]_{y=Y} dx.$$

If $f_{yy} > 0$ and $f_{y'y'} > 0$ what is the condition that $Y(x)$ is a minimizer for the functional I?

The Euler-Lagrange equation in these co-ordinates is

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0\,.$$

[**5**] Use spherical co-ordinates

$$x = \sin\theta\cos\phi; \qquad y = \sin\theta\sin\phi; \qquad z = \cos\theta$$

to show that the length of an arc between two points on the surface of a sphere of radius unity is

$$s = \int_{\theta_1}^{\theta_2} \left[1 + \sin^2\theta \left(\frac{d\phi}{d\theta}\right)^2\right]^{1/2} d\theta\,.$$

Hence show that the geodesic connecting these points is given by

$$a\sin(\phi + \delta) + \cot\theta = 0$$

where δ and a are arbitrary constants. Show that this represents the arc of a circle formed by the intersection of the sphere with a plane that passes through both of these points and the centre of the sphere.

The Euler-Lagrange equation in these co-ordinates is

$$\frac{\partial f}{\partial\phi} - \frac{d}{d\theta}\left(\frac{\partial f}{\partial\phi'}\right) = 0\,.$$