M2A1 – 2003

[**1**] Consider the pair of ordinary differential equations in the variables $x(t)$ and $y(t)$

$$\dot{x} = x - x^2 - xy\,, \qquad \dot{y} = \frac{3}{2}y - y^2 - 2xy\,.$$

Show that this system has four fixed points; one saddle, two stable nodes and an unstable node. Sketch the phase plane, marking regions where dy/dx is postive, negative, zero or infinity.

[**2**] Show that the pair of ordinary differential equations

$$\dot{x} = y(1+y^2)\,, \qquad \dot{y} = 1 - x^2.$$

has one centre and one saddle in the phase plane. Sketch the phase plane diagram.

By direct integration of the equation for dy/dx, confirm that a family of periodic orbits surround the centre.

[**3**] A string of mass per unit length (density) ρ is clamped at two points $x = 0$ and $x = L$ and stretched between them with a tension T. Show that *small* lateral oscillations in the y-direction are governed by the wave equation

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} = 0\,,$$

where $c^2 = T/\rho$, explaining any approximations you might use. Given the end conditions, use the method of separation of variables to show that the general solution is

$$y(x,t) = \sum_{n=0}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left\{A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right)\right\}\,,$$

where A_n and B_n are arbitrary constants.

[**4**] Traffic of density $\rho(x,t)$ flows on a motorway with no entry or exit points. $\rho(x,t)$ and the flux $Q(\rho)$ obey the conservation equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial Q(\rho)}{\partial x} = 0\,.$$

For the model to be effective, $Q(\rho)$ must be chosen such that shocks occurring in ρ should travel backwards. Discuss what general shape the flux $Q(\rho)$ should take, given that the initial distribution of traffic on the motorway $\rho(x,0) = f(x)$ is Gaussian-shaped. If the flow velocity is given by $V = Q/\rho$ and the propagation velocity by $c(\rho) = dQ/d\rho$, show that your choice is consistent with $V > c$.

If $Q = \rho(\rho_0 - \frac{1}{2}\rho)$, with ρ_0 a positive constant, show that this choice has the right properties for a traffic flow model. Solve for $\rho(x,t)$ when initial data $\rho(x,0) = f(x)$ is given by

$$f(x) = \begin{cases} x & 0 \le x \le 1 \\ 2 - x & 1 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

At what time does the shock occur?

[**5**] In cylindrical co-ordinates (r, θ, z) show that a small element of arc length ds is given by

$$(ds)^2 = (dr)^2 + r^2(d\theta)^2 + (dz)^2\,.$$

Show that the arc length of a curve on the cylindrically symmetric parabolic surface $z = \frac{1}{2}r^2$, with end points represented by θ_1 and θ_2, is given by

$$\text{arc length} = \int_{\theta_1}^{\theta_2} ds = \int_{\theta_1}^{\theta_2} \left\{(1+r^2)r'^2 + r^2\right\}^{1/2} d\theta$$

where $r' = dr/d\theta$.

Show that the arc length takes stationary values when θ and $r = c\cosh\alpha$ satisfy the relation

$$\int \left(\text{sech}^2\alpha + c^2\right)^{1/2} d\alpha = \theta + \delta\,,$$

where $c > 0$ and δ are constants.

You may use the Euler-Lagrange equation in the form

$$f_\theta + \frac{d}{d\theta}\left(r' f_{r'} - f\right) = 0.$$