M2A1 – 2002

[**1**] An SIR disease model relating the number of susceptibles $x(t)$, infectives $y(t)$ and recovered $z(t)$, is governed by the set of ordinary differential equations

$$\dot{x} = -x^2 y + \alpha \qquad \dot{y} = x^2 y - y \qquad \dot{z} = y - \alpha$$

where $x + y + z = N$ and the birthrate constant α is positive.

Show that there is only one critical point in the first quadrant of the $x - y$ phase plane and that this is a stable node when $\alpha > 2$ and a stable spiral when $\alpha < 2$. What is the classification of the critical point when $\alpha = 2$?

When α lies in the range $0 < \alpha < 2$, sketch some typical orbits in the first quadrant of the phase plane, marking the sign of dy/dx in appropriate regions.

What is the final value of z? Comment on the physical significance of these results for both ranges of α.

[**2**] (a) Construct the phase plane diagram for the nonlinear oscillator equation

$$\ddot{x} + x - x^5 = 0,$$

using the phase plane in which $y = \dot{x}$. The dot denotes differentiation with respect to time t.

Show that the separatrices are given by

$$3y^2 = (x^2 - 1)^2(x^2 + 2),$$

and sketch these curves on your diagram.

(b) Show that the nonlinear oscillator equation

$$\ddot{x} + x + x^5 = 0,$$

has just one critical point and that it is a centre. Determine an energy equation for the family of trajectories. Express this in polar coordinates, given by $x = r\cos\theta, y = r\sin\theta$. By considering the roots of the cubic equation for $\lambda = r^2$ (or otherwise) show that on any trajectory, r is a bounded, periodic and single-valued function of θ. From this information, can you deduce what property these trajectories have?

[**3**] The wave equation

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} = 0.$$

governs small lateral oscillations on a string of length π which is fixed at $x = 0$ and $x = \pi$. Given the end conditions, use the method of separation of variables to find the general solution.

The string is released at $t = 0$ with a shape profile

$$y(x, 0) = x(\pi - x)$$

and zero velocity. Show that for $t \geq 0$

$$y(x,t) = \frac{8}{\pi}\sum_{m=0}^{\infty}\frac{\sin[(2m+1)x]\cos[(2m+1)ct]}{(2m+1)^3}.$$

Deduce that for all time the maximum displacement of the string remains at its midpoint.

[**4**] The density $\rho(x, t)$ of a quantity evolves according to the kinematic wave equation

$$\frac{\partial \rho}{\partial t} + c(\rho)\frac{\partial \rho}{\partial x} = 0$$

where $c(\rho)$ is the propagation velocity.

Show that for initial data $\rho(x, 0) = f(x)$ there is an implicit solution of the form

$$\rho = f\left(x - c(\rho)t\right).$$

Consider the set of initial conditions:

$$f(x) = \begin{cases} x & 0 \leq x \leq 1 \\ 1 & 1 \leq x \leq 2 \\ 3 - x & 2 \leq x \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

When (i) $c(\rho) = \rho$ and (ii) $c(\rho) = 2 - \rho$, find the exact solution $\rho = \rho(x, t)$ in each case. Show that a shock develops at time $t = 1$ in each case. Sketch the evolution of ρ for the four times $t = 0,\ 0 < t < 1,\ t = 1$ and $t > 1$ in case (i).

[**5**] (a) For a variational problem of the form

$$I = \int_{\theta_1}^{\theta_2} f\left[\theta,\, x(\theta),\, x'(\theta)\right]\, d\theta,$$

where $x' = dx/d\theta$, if $x(\theta)$ satisfies the Euler-Lagrange equation

$$\frac{\partial f}{\partial x} - \frac{d}{d\theta}\left(\frac{\partial f}{\partial x'}\right) = 0$$

then I takes stationary values. Show that the Euler-Lagrange equation can be re-written in the form

$$\frac{d}{d\theta}\left(x'\frac{\partial f}{\partial x'} - f\right) + \frac{\partial f}{\partial \theta} = 0.$$

(b) A pair of cones, joined at their tips at the origin, have the x-axis as their principal axes of symmetry. Their curved surfaces are represented by

$$y^2 + z^2 = x^2$$

which is parametrized by $y = x\cos\theta$ and $z = x\sin\theta$. Show that the arc length of a curve on one of these surfaces, with end points represented by θ_1 and θ_2, is given by

$$\text{arc length} = \int_{\theta_1}^{\theta_2} ds = \int_{\theta_1}^{\theta_2} \left\{2\left(\frac{dx}{d\theta}\right)^2 + x^2\right\}^{1/2} d\theta.$$

Show that the arc length takes stationary values when x and θ satisfy the differential equation

$$2c^2\left(\frac{dx}{d\theta}\right)^2 = x^2\left(x^2 - c^2\right)$$

where c is a constant. Show that this differential equation is satisfied by

$$x = \pm c\,\text{cosec}\left(\frac{\theta + \delta}{\sqrt{2}}\right)$$

where δ is another constant.