Imperial College London

BSc and MSci EXAMINATIONS (MATHEMATICS)

January 2007

M1S (Test)

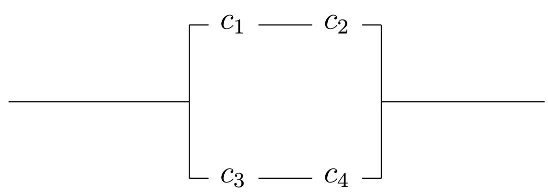
Probability and Statistics

- Affix ONE label to each answer book that you use. DO NOT use the label with your name on it.
- Write your answers in a single answer book, using continuation books if necessary.
- Credit will be given for all questions attempted, but extra credit will be given for complete or nearly complete answers.
- The question in Section A will be worth $1\frac{1}{2}$ times as many marks as either question in Section B.
- Calculators may not be used.

SECTION A

1. (i) Events A, B and C each have probability $\frac{1}{2}$. Also, B and C are independent, and A and C are exclusive. Using a set diagram, or otherwise, calculate $\mathrm{P}(B \cap C)$, $\mathrm{P}(A \cup C)$, $\mathrm{P}(C \mid B)$ and $\mathrm{P}(A \cap B)$.

(ii) The diagram below shows a system of independently-operating components, $c_1, \ldots, c_4$, in which component c_j has reliability p_j $(=$ probability of operating satisfactorily). Calculate the reliability of the system.



(iii) The random variables X_1 and X_2 are independent with distribution functions $F_1(x)$ and $F_2(x)$, respectively. Write down the forms of the distribution functions of $V = \max(X_1, X_2)$ and $U = \min(X_1, X_2)$ in terms of F_1 and F_2.

(iv) The random variable Y takes values $1, \ldots, m$ each with probability m^{-1}. Derive its probability generating function.

(v) A point is chosen uniformly at random on a line of length l, and R is the ratio of lengths of the line segments so produced. Obtain the distribution function of R.

(vi) The random variables X_1 and X_2 are independent, and each takes values 0,1,2 with probabilities p_0, p_1, p_2, where $p_0 + p_1 + p_2 = 1$. Write down the probability distribution of $\mid X_1 - X_2 \mid$.

SECTION B

2. (a) The probability distribution of the discrete random variable X is specified as $\mathrm{P}(X = j) = k/j$ for $j = 1, 2, ..., m$. Show that X has mean mk and obtain its variance in terms of m and k. Given that its mean is $\frac{4}{3}$ what is the numerical value of its variance? (You may assume that $\sum_{j=1}^{m} j = \frac{1}{2}m(m+1)$.)

   (b) Suppose that the continuous random variable Y has density $ly(3-y)$ on (0,3). Determine the constant l, calculate the mean and variance of Y, and calculate the mean of $\{Y(3-Y)\}^{-1}$.

3. (a) The random variable Y_1 has mean 5 and variance 3; Y_2 has mean –4 and variance 4; Y_1 and Y_2 have covariance 2. Calculate the mean and variance of $2Y_1 - 3Y_2$ and the covariance between $Y_1 + Y_2$ and $Y_1 - Y_2$.

   (b) The discrete random variable X takes values 1,2 with probabilities $\frac{3}{4}, \frac{1}{4}$. Conditionally on $X = x$, the continuous random variable Y has density function $x\mathrm{e}^{-xy}$ on $(0, \infty)$. Calculate $\mathrm{P}(Y > y \mid X = x)$ in terms of x, and obtain $\mathrm{P}(Y > y)$. Given that $\mathrm{E}(Y \mid X = x) = x^{-1}$, calculate $\mathrm{E}(Y)$.