Imperial College London

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2008

This paper is also taken for the relevant examination for the Associateship.

M1S

Probability and Statistics

Date: Thursday, 15th May 2008 Time: 10 am – 12 noon

Answer all questions. Each question carries equal weight.

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. (i) A candidate, say C, is required to choose one answer out of m alternatives on a question. Suppose that the probability that C knows the correct answer is p, in which case C will choose it; otherwise C will choose randomly. Given that the correct answer was selected, what is the probability that C knew it?

(ii) Consider successive offspring: define events
$G_1 = \{$*first child is a girl*$\}$ and $G_2 = \{$*second child is a girl*$\}$, and let
$p = \mathrm{P}(G_1)$, $q = \mathrm{P}(G_2 \mid G_1)$ and $r = \mathrm{P}(G_2 \mid \overline{G}_1)$.
Calculate the probabilities of $\{$*girl first, girl second*$\}$, $\{$*girl first, boy second*$\}$, $\{$*boy first, girl second*$\}$ and $\{$*boy first, boy second*$\}$ in terms of p, q and r.

(iii) An erratic darts player hits the circular board area uniformly, $i.e.$ with constant probability density over its area. Calculate p, the probability that a dart (that hits the board) lands closer to the centre than to the edge of the board. Given three throws that hit the board, what is the probability of landing two or more darts closer to the centre than to the edge of the board?

(iv) Random variable X takes values $1, 2, 3$ with probabilities $\frac{1}{6}, \frac{1}{3}, \frac{1}{2}$. A second random variable, Y, depends on X in the following way: given that $X = x$, Y takes values x, $2x$, $3x$ with probabilities $\frac{2}{5}, \frac{1}{5}, \frac{2}{5}$.

Show that $\mathrm{E}(Y \mid X = x) = 2x$ and hence calculate $\mathrm{E}(Y)$.

2. (i) A cube is constructed whose side Y is random with density function $f_Y(y) = 2y^{-2}$ on $(1, 2)$. Calculate its expected volume and show that the probability that its volume exceeds 4 is $2^{1/3} - 1$.

(ii) Given that $G(u) = k(u+2)^3$ is the pgf of a discrete random variable Z, list the possible values of Z and the corresponding probabilities. Show that Z can be expressed as the sum of three independent random variables with the same distribution and identify this distribution. Any clearly stated property of the pgf may be assumed without proof.

(iii) Seeds are generated randomly so that their number, M, has a geometric distribution with $\mathrm{P}(M = m) = (1-p)^{m-1}p$ $(m = 1, 2, \ldots;\ 0 < p < 1)$. Germination of any seed occurs with probability q independently of other seeds $(0 \le q \le 1)$. Calculate the probability that the number of germinated seeds is 0, and the probability that the number of germinated seeds is 1.
You may assume the identities
$\sum_{m=1}^{\infty} x^{m-1} = \frac{1}{1-x}$ and $\sum_{m=1}^{\infty} m x^{m-1} = \frac{1}{(1-x)^2}$ for $0 < x < 1$.

3. Two chains support a weight and if either breaks the weight will fall. The times to breakage of the chains, T_1 and T_2, are independent random variables each with density function $\lambda \mathrm{e}^{-\lambda t}$ on $(0, \infty)$, with $\lambda > 0$.

(i) Obtain an expression for $\mathrm{P}(T_1 > t)$.

Let T be the time at which the weight falls.

(ii) Determine $\mathrm{P}(T > t)$ and hence find the density function of T.

(iii) How is the median of T related to that of T_1?

(iv) Calculate the upper limit for λ needed to ensure that there is at least a 90% chance that the weight will remain supported for a length of time not less than l.

4. The random lifetime of a system is Y and the random stress on it is X. Given that $X = x$, Y has survivor function $\mathrm{P}(Y > y \mid X = x) = \mathrm{e}^{-xy}$ on $(0, \infty)$; further, X has survivor function $\mathrm{P}(X > x) = \mathrm{e}^{-\lambda x}$ on $(0, \infty)$, with $\lambda > 0$.

(i) Calculate the conditional density function $f_Y(y \mid X = x)$, and derive $\mathrm{E}(Y \mid X = x)$.

(ii) Use the identity

$$\mathrm{P}(Y > y) = \int_0^\infty \mathrm{P}(Y > y \mid X = x) f_X(x) \, \mathrm{d}x$$

to verify that Y has unconditional survivor function $\lambda/(\lambda + y)$.

(iii) What lifetime should the manufacturer of the system guarantee so that only 5% of systems fail before this guaranteed time?

(iv) Calculate the mean lifetime of systems that fail before this guarantee time runs out.