Imperial College London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2007

This paper is also taken for the relevant examination for the Associateship.

M1S

Probability and Statistics

Date: Monday, 14th May 2007 Time: 2 pm – 4 pm

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

1. (a) Events E_1 and E_2 are exclusive, and E_2 and E_3 are independent. Let $\mathrm{P}(E_j) = p_j$ ($j = 1, 2, 3$) and $\mathrm{P}(E_1 \mid E_3) = p_{13}$. Express $\mathrm{P}(E_1 \mid E_2)$, $\mathrm{P}(E_2 \mid E_3)$, $\mathrm{P}(E_1 \cap E_3)$, $\mathrm{P}(E_3 \mid E_1)$ and $\mathrm{P}(E_1 \cup E_3)$ in terms of the ps.

(b) A machine component is subject to three types of corrosion, C_1, C_2 and C_3. The known probabilities are $\mathrm{P}(C_1) = 0.1$, $\mathrm{P}(C_2) = 0.05$, $\mathrm{P}(C_3) = 0.01$, and $\mathrm{P}(C_1 \cup C_2) = 0.12$. It is also known that C_1 and C_3 occur independently, and that C_2 and C_3 are exclusive. Compute the probabilities of the following states of deterioration:

(i) C_1 and C_3 both present;

(ii) C_1 and C_2 both present;

(iii) C_1, C_2 and C_3 all present;

(iv) C_2 present, given that C_1 is present;

(v) C_1 present, given that C_3 is present.

2. (a) A man can take either of two buses to work. The waiting times for them are T_1 and T_2, which are independent each with density $\lambda e^{-\lambda t}$ on $(0, \infty)$. Show that $\mathrm{P}(T_1 > w) = e^{-\lambda w}$. Assuming that he will take the first bus to arrive, what is the probability that his waiting time, say W, will exceed w?

(b) Two generators operate independently and the cost of maintenance for each is c per month. The probability of breakdown during any given month is p for a maintained generator and 1 for an unmaintained generator. The cost of being without a generator, *i.e.* when neither generator is operating, is b. Calculate the expected costs over a given month of the following strategies: S0 – maintain neither generator; S1 – maintain just one generator; S2 – maintain both generators. Given that $p(1-p) < \frac{c}{b} < 1 - p$, which is the cheapest strategy?

3. (a) Express $a = \mathrm{E}(Y^2)$ and $b = \mathrm{E}\{Y(1-Y)\}$ in terms of $\mu = \mathrm{E}(Y)$ and $\sigma^2 = \mathrm{var}(Y)$. Evaluate a and b numerically when Y is uniformly distributed on (0,1).

(b) The continuous random variable X has density $f(x) = (\alpha+1)x^\alpha/l^{\alpha+1}$ on $(0, l)$. Obtain the distribution function, $F_X(x)$, of X. Calculate $\mathrm{E}(X)$ and determine the distribution of the random variable $F_X(X)$.

4. A river level fluctuates randomly, the height V at any instant having probability density $f(v) = \xi^{-1}(1 + v/\xi)^{-2}$ on $(0, \infty)$ with $\xi > 0$.

(i) Find the distribution function of V and calculate the median river height.

(ii) Evaluate $\mathrm{P}(V > a + b \mid V > a)$, where $0 < a < b$.

Now suppose that the height is recorded at the same time on n successive days, producing independent readings $v_1, ..., v_n$.

(iii) Calculate the probability that all n heights lie in the range (a, b).

(iv) For the case $n = 4$ and $\xi = 3$ show that the probability that at most one reading falls below the level $a = 1$ is $7 \times 3^3/4^4$.

5. The random variable X_1 takes values –1,0,1 with probabilities $\frac{1}{4}, \frac{1}{2}, \frac{1}{4}$. If $X_1 = 0$, X_2 takes values 0,1 with probabilities $\frac{1}{2}, \frac{1}{2}$; otherwise, it takes values –1,0 with probabilities $\frac{1}{2}, \frac{1}{2}$.

(i) Show that $\mathrm{P}(X_1 = -1, X_2 = 0) = \frac{1}{8}$. Draw up a table showing the joint distribution of X_1 and X_2.

(ii) Derive the marginal distribution of X_2 and calculate its mean and variance.

(iii) Determine the distribution of $X_1 + X_2$.

(iv) Evaluate $\mathrm{E}(X_2 \mid X_1 = x)$ for $x = -1, 0, 1$, and derive $\mathrm{E}(X_2)$ from these conditional means.