Imperial College London

UNIVERSITY OF LONDON

BSc and MSci EXAMINATIONS (MATHEMATICS)

May-June 2006

This paper is also taken for the relevant examination for the Associateship.

M1S

Probability and Statistics I

Date: Thursday, 11th May 2006 Time: 2 pm – 4 pm

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used.

Statistical tables will not be available.

1. (a) Show that:

   (i) if $E_1 \subseteq E_2$ and $\mathrm{P}(E_2) > 0$, then $\mathrm{P}(E_1 \mid E_2) \geq \mathrm{P}(E_1)$;

   (ii) if $\mathrm{P}(E_1) = 0$ and $\mathrm{P}(E_2) > 0$, then $\mathrm{P}(E_1 \mid E_2) = 0$.

   (b) Three beads are drawn blind from a bag containing 3 red beads and 2 blue beads. What is the probability that exactly 2 of the drawn beads are red when the drawing is done:

   (i) with replacement?

   (ii) without replacement?

   (iii) with replacement if the first bead drawn is red, otherwise without replacement?

2. The random variable X has density function $6x(1-x)$ on (0,1).

   (i) Calculate the mean of X, the mean of X^2 and the variance of X.

   (ii) Let $Y = X(1-X)$. What range of values can Y take? Calculate the mean of Y, and the covariance of Y/X with $Y/(1-X)$.

   (iii) Let

$$Z = \left(X - \frac{1}{2}\right)^2.$$

   Show that the distribution function of Y can be expressed as $F_Y(y) = \mathrm{P}\left(Z \geq \frac{1}{4} - y\right)$. Write down the probability distribution of $Y + Z$. Why are Y and Z perfectly negatively correlated?

   (Give your answers in (i) and (ii) as fractions; there is no need to express them in decimal form.)

3. (a) The random variable Z has standard normal distribution N(0,1). Find the probability that the equation $x^2 - 2Zx + 1 = 0$ has no real solution. How is this probability changed if it is to be conditional on $Z > 0$? (You may assume that $\Phi(1) = 0.8413$.)

   (b) A sequence of independent trials is performed in each of which the outcome value is 1 or 0 and $\mathrm{P}(outcome = 1) = q$. Let R be the trial number on which the outcome '1' first occurs. Write down an expression for $p_r = \mathrm{P}(R = r)$ and calculate $\mathrm{E}(R)$. Suppose now that q is itself generated randomly, taking values $\frac{1}{4}, \frac{3}{4}$ with respective probabilities $\frac{1}{2}, \frac{1}{2}$. What form does p_r take now? How is $\mathrm{E}(R)$ affected?

   Hint: $\sum_{r=1}^{\infty} r\lambda^{r-1} = (1-\lambda)^{-2}$ for $0 < \lambda < 1$.

4. (a) The distribution of the random variable X is given by $\mathrm{P}(X=1)=\frac{1}{3}$ and $\mathrm{P}(X=2)=\frac{2}{3}$. A second random variable, Y, is dependent on X: if $X=1$, Y takes values 0,1,2 with probabilities $\frac{1}{2},\frac{1}{4},\frac{1}{4}$; if $X=2$, Y takes values 1,2,3 with respective probabilities $\frac{1}{3},\frac{1}{3},\frac{1}{3}$. Draw up a table showing the joint distribution of (X,Y). Determine the marginal distribution of Y and the conditional distribution of X given that $Y=1$.

(b) The joint density function of (U,V) is

$$f(u,v)=k(1-u-v+uv) \quad \text{on} \quad (0,1)^2.$$

Determine the constant k and the marginal densities of U and V.
Calculate the covariance of U and V. Are U and V independent?

5. The random variable X takes values 0,1,2 with probabilities $\frac{1}{4},\frac{1}{2},\frac{1}{4}$. Evaluate its mean and variance, and its probability generating function (pgf). Hence verify that X may be expressed as a sum, Y_1+Y_2, of two independent, identically distributed random variables, and specify their common distribution. Write down the pgf of $S_n=X_1+\ldots+X_n$, where the X_i are independent and all distributed as X.
Hence show that

$$\mathrm{P}(S_n=r)=2^{-2n}\binom{2n}{r}$$

for a certain range of r-values, which you should identify.
Use the pgf of S_n to calculate its mean and variance.

You may assume the binomial expansion:

$$(a+b)^m=\sum_{j=0}^{m}\binom{m}{j}a^j b^{m-j}.$$