Imperial College London

Department of Mathematics

BSc and MSci EXAMINATIONS (MATHEMATICS) MAY–JUNE 2004

This paper is also taken for the relevant examination for the Associateship.

M1S Probability and Statistics I

DATE: Friday, 14th May 2004 TIME: 10 am – 12 noon

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used. Statistical tables will not be available.

1.

(a) State the three axioms of probability for events defined on a sample space Ω.

(b) Using the axioms of probability:

(i) Prove that for any events $E, F \subseteq \Omega$, the probability of *one and only one* of them occurring is

$$\mathbf{P}(E) + \mathbf{P}(F) - 2\mathbf{P}(E \cap F).$$

(ii) A box contains 100 balls numbered serially from 1 to 100. A ball is drawn at random. Calculate the probability that the number on the ball is divisible by one and only one of the primes 3 and 5.

(c) If $\mathbf{P}(E|F) > \mathbf{P}(E)$ and $\mathbf{P}(E) > 0$ and $\mathbf{P}(F) > 0$, prove that

(i) $\mathbf{P}(F|E) > \mathbf{P}(F)$,

(ii) $\mathbf{P}(E|F') < \mathbf{P}(E)$.

2.

(a) Suppose that $F_1, \ldots, F_n$ form a partition of the sample space Ω, and $\mathbf{P}(F_i) > 0$ for $i = 1, \ldots, n$. Let E be any event in Ω, with $\mathbf{P}(E) > 0$.

(i) State the Theorem of Total Probability.

(ii) Derive Bayes' formula for $\mathbf{P}(F_i|E)$.

(b) 10% of computer hard disks produced by a manufacturer are faulty. A method has been designed to test whether the disks are faulty or not. This test has a probability of 0.9 of giving a positive result when applied to a faulty disk, and a probability of 0.1 of giving a positive result when applied to a perfect disk. A disk is chosen at random and tested.

(i) What is the probability that the test gives a positive result?

(ii) Given a positive result, what is the probability the disk is faulty?

(c) A bag contains n counters labelled with numbers $1, 2, \ldots, n$ with $n \geq 2$. Two counters are drawn at random, one after the other, without replacement of the first counter. Carefully define the sample space for this experiment. Find the probability that

(i) the sum of the numbers on the counters is less than 6,

(ii) the numbers on the two counters differ by two.

[*Hint: For both* **(c)(i)** *and* **(c)(ii)** *look at the cases* $n = 2, 3$, *etc separately until you see the general pattern.*]

3.

The discrete random variable X has probability mass function $\mathbf{P}(X = x) = c/x$ for $x = 1, 2, \dots, n$, where c and n are constants.

(a) Express c in terms of n.

(b) Express $E_{f_X}[X]$ and $E_{f_X}[X(X-1)]$ in terms of c and n.
[*Recall:* $\sum_{j=1}^{n} j = n(n+1)/2$.]

(c) If $E_{f_X}[X] = 4/3$ show that $n = 2$ and deduce the value of c. [*You may assume without proof that* $n = 2$ *is a unique solution.*]

(d) Using the values of c and n from part **(c)**,

- **(i)** Show that we can write $\mathbf{P}(X = x)$ in the form $\mathbf{P}(X = x) = \theta^{x-1}(1-\theta)^{2-x}$, and state the range of X and the value of θ.
- **(ii)** Derive the corresponding probability generating function and use it to calculate $E_{f_X}[X(X-1)]$; does the result agree with that obtained in part **(b)**?
- **(iii)** How does the probability mass function derived here differ from that of a Bernoulli random variable?

4. Suppose X and Y are two independent random variables. Let $X \sim \text{Gamma}\,(n, \nu)$, with n a positive integer and ν a positive constant, and probability density function

$$f_X(x) = \frac{\nu^n}{(n-1)!} x^{n-1} e^{-\nu x} \quad (x > 0),$$

and $Y \sim \text{Exponential}\,(\nu)$ with probability density function

$$f_Y(y) = \nu e^{-\nu y} \quad (y > 0).$$

The probability distribution of $Z = X + Y$ is given by the convolution

$$f_Z(z) = \int_0^z f_X(x) f_Y(z - x)\,dx.$$

(a) Show that $Z \sim \text{Gamma}\,(n+1, \nu)$.

(b) Use the result in **(a)** to carefully deduce the distribution of the sum of n independent random variables, each having the $\text{Exponential}\,(\nu)$ distribution.

(c) Derive the moment generating function of Y.

(d) Use the results of parts **(b)** and **(c)** to calculate the variance of X.

5. Suppose X is a continuous random variable with a probability density function that is symmetric about zero. Let $Y = |X|$.

(a) Show that the cumulative distribution function of Y, namely $F_Y(y)$, is related to the cumulative distribution function of X, namely $F_X(x)$, via

$$F_Y(y) = \begin{cases} 2F_X(y) - 1, & \text{if } y \geq 0; \\ 0, & \text{if } y < 0. \end{cases}$$

(b) Let X be a continuous random variable having the Laplace (also called the double exponential) probability density function

$$f_X(x) = \frac{1}{2} \exp\left(-|x|\right), \qquad -\infty < x < \infty.$$

(i) Find $F_X(x)$ for $x \geq 0$.

(ii) Hence derive $F_Y(y)$ and $f_Y(y)$.

(iii) Show that $E_{f_Y}[Y] = 1$.