Imperial College London

Department of Mathematics

BSc and MSci EXAMINATIONS (MATHEMATICS) MAY–JUNE 2003

This paper is also taken for the relevant examination for the Associateship.

M1S Probability and Statistics I

DATE: Friday, 16th May 2003 TIME: 10 am – 12 noon

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Calculators may not be used. Statistical tables will not be available.

1.

(a) State the three axioms of probability for events defined on a sample space Ω.

Use the axioms to prove that for any events $E, F \subseteq \Omega$,

(i) $\mathbf{P}(E' \cap F) = \mathbf{P}(F) - \mathbf{P}(E \cap F)$.

(ii) $E \subseteq F \Longrightarrow \mathbf{P}(E) \leq \mathbf{P}(F)$.

(b) Use induction to prove that for arbitrary events $E_1, \ldots, E_n \subseteq \Omega$,

$$\mathbf{P}\left(\bigcup_{i=1}^{n} E_i\right) \leq \sum_{i=1}^{n} \mathbf{P}(E_i).$$

(c) Using De Morgan's Laws deduce that

$$\mathbf{P}\left(\bigcap_{i=1}^{n} E_i\right) \geq \sum_{i=1}^{n} \mathbf{P}(E_i) - (n-1).$$

2.

(a) A hand of 13 cards is dealt from a normal shuffled pack of 52. If the hand contains exactly two kings, what is the probability that it contains exactly one ace?
[Give the answer in the form $m_1\binom{m_2}{m_3}/\binom{m_4}{m_5}$ for integers $m_1, \ldots, m_5$].

(b) A random number N of dice are thrown. Assume that $\mathbf{P}(N = i) = 2^{-i}$, $i \geq 1$. The sum of the scores is S.

(i) Evaluate $\mathbf{P}(S = 4|N = i)\mathbf{P}(N = i)$ for $i = 1, 2, 3$ and 4.
[Give the answers as fractions of the form $1/n$, for integer n.]

(ii) Show that the probability that $S = 4$ given N is odd, can be written

$$\frac{\mathbf{P}(S = 4|N = 1)\mathbf{P}(N = 1) + \mathbf{P}(S = 4|N = 3)\mathbf{P}(N = 3)}{\sum_{j=0}^{\infty} 2^{-(2j+1)}}.$$

(iii) Evaluate this probability.
[Give the answer as a fraction].

3.

(a) (i) For a discrete random variable X with range $\mathbb{X} = \{x_1, x_2, \ldots\}$ and probability mass function f_X the variance of X may be written

$$\mathrm{var}_{f_X}[X] = \sum_{x_i \in \mathbb{X}} (x_i - \mu_X)^2 f_X(x_i),$$

where $\mu_X = E_{f_X}[X]$. Starting with this expression show that it is also possible to write the variance as

$$\begin{aligned} \mathrm{var}_{f_X}[X] &= E_{f_X}[X^2] - (E_{f_X}[X])^2 \\ \text{and} \quad \mathrm{var}_{f_X}[X] &= E_{f_X}[X(X-1)] + E_{f_X}[X] - (E_{f_X}[X])^2. \end{aligned}$$

(ii) Name the components $E_{f_X}[X]$, $E_{f_X}[X(X-1)]$ and $E_{f_X}[X^2]$ in terms of moments and factorial moments.

(b) A random variable Y has the Poisson distribution with parameter λ, i.e., $Y \sim \text{Poisson}(\lambda)$ if it has probability mass function

$$\mathbf{P}(Y = j) = \frac{\lambda^j e^{-\lambda}}{j!}, \; j = 0, 1, \ldots\ldots \quad \text{and } \lambda > 0.$$

Suppose a random variable X has probability mass function

$$\mathbf{P}(X = j) = \sum_{n=j}^{\infty} \binom{n}{j} p^j (1-p)^{n-j} \frac{\lambda^n e^{-\lambda}}{n!}, \; j = 0, 1, \ldots \quad \text{and } 0 < p < 1.$$

(i) Show that $X \sim \text{Poisson}(\lambda p)$, i.e., the parameter for X is λp compared to just λ for Y. [*Hint: Note carefully the summation bounds*].

(ii) Derive the probability generating function, or factorial moment generating function, $G_X(t)$ of X.

(iii) Use $G_X(t)$ to find the mean and variance of X.

4.

Let the positive random variable X represent the lifetime of an electrical system. The system consists of two components connected in parallel so that $X = \max\{X_1, X_2\}$ where X_1 and X_2 are independent lifetimes of component 1 and component 2, respectively both having exponential failure time distributions, i.e., $f_{X_1}(x) = \lambda_1 e^{-\lambda_1 x},\ x \geq 0$, and $f_{X_2}(x) = \lambda_2 e^{-\lambda_2 x},\ t \geq 0$, with $\lambda_1, \lambda_2 > 0$.

(a) Find the cumulative distribution function, $F_X(x)$, of X.
[*Hint: for random variables* U *and* V, $\mathbf{P}(\max\{U, V\} \leq w) =$
$\mathbf{P}(U \leq w \cap V \leq w)$.]

(b) Hence show that the probability density function $f_X(x)$ of X is given by

$$f_X(x) = \lambda_1 e^{-\lambda_1 x} + \lambda_2 e^{-\lambda_2 x} - (\lambda_1 + \lambda_2) e^{-(\lambda_1+\lambda_2)x},\ x \geq 0,$$

and comment on the form of this function.

(c) Demonstrate that $f_X(x)$ does indeed integrate to unity, as required of any probability density function.

(d) Derive the moment generating function for X and hence find the second moment $E_{f_X}[X^2]$ of X.

5.

(a) (i) Suppose that the random variable X is uniformly distributed over the interval $[a, b]$. Write down the cumulative distribution function for X.

(ii) Use the cumulative distribution function to obtain the probability density function (with range $\mathbb{Y}$) for $Y = e^X$ when $a = -2$ and $b = 2$.

(iii) When $a = 0$ and $b = 5$ show that the probability is $3/5$ that the roots of the equation

$$4y^2 + 4Xy + X + 2 = 0$$

are both real.

(b) Let Z be normally distributed with zero mean, unit variance, and cumulative distribution function $\Phi(\cdot)$.

(i) Write down the cumulative distribution function of $Y = Z^2$ in terms of $\Phi(\cdot)$.

(ii) Hence find the probability density function of Y.

(iii) Show that $E_{f_Y}[Y] = 1$. Why is this result obvious directly?