M1P2 ALGEBRA I, May 2008

1. Which of the following statements are true and which are false ? For those which you think are true, give a brief justification, and for those you think are false, give a counterexample.

(i) Every abelian group is cyclic.

(ii) The number of subgroups of S_5 of size 3 is equal to 10.

(iii) S_{14} has an element of order 44.

(iv) $GL(2, \mathbb{R})$ has an element of order 3.

(v) $\mathbb{Z}_{11}^*$ is a cyclic group.

(vi) There is a non-abelian group G such that $x^6 = e$ for all $x \in G$.

(vii) There is no group G such that $|G| = 12$ and $x^5 = e$ for all $x \in G$.

2. (a) Let G be a finite group, let H be a subgroup of G, and let $x \in G$. Define what is meant by the right coset Hx.

Prove that if $x, y \in G$ then either $Hx = Hy$ or $Hx \cap Hy = \emptyset$.

Now let $G = S_3$ and let H be the cyclic subgroup of G generated by the 2-cycle $(1\,2)$. List all the distinct right cosets of H in G.

(b) Find all integers between 2 and 50 which divide $9^{11} - 1$.

3. (a) Let V be a vector space, and let $\{v_1, \ldots, v_k\}$ be a set of vectors in V. Define what is meant by the following statements:

$\{v_1, \ldots, v_k\}$ is a *linearly independent* set

$\{v_1, \ldots, v_k\}$ is a *spanning set* for V

$\{v_1, \ldots, v_k\}$ is a *basis* of V.

(b) Suppose $\{v_1, \ldots, v_k\}$ is a spanning set for V, and let v be a non-zero vector in V. Prove that there is a value of i such that the set

$$\{v, v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_k\}$$

is a spanning set for V. (*You may NOT use any results from the lectures unless you prove them.*)

(c) Let V be the vector space of polynomials over $\mathbb{R}$ of degree at most 3, and let v_1, v_2, v_3, v_4 be the following vectors in V:

$$\begin{aligned} v_1 &= x - x^3 \\ v_2 &= -1 + x^2 + 3x^3 \\ v_3 &= x + x^2 + 2x^3 \\ v_4 &= x^3 \end{aligned}$$

Show that $\{v_1, v_2, v_3, v_4\}$ is a spanning set for V.

If $v = 2 + 3x + x^2$, find a value of i as in (b).

4. Let V, W be vector spaces. Define what is meant by a linear transformation $T : V \to W$, and by the kernel $\mathrm{Ker}(T)$ of T.

Prove that $\mathrm{Ker}(T)$ is a subspace of V.

Find the dimension of $\mathrm{Ker}(T)$ in the following examples, justifying your answers:

(i) $T : \mathbb{R}^4 \to \mathbb{R}^3$ defined by

$$T(x_1, x_2, x_3, x_4) = (x_1 - x_3 + 2x_4,\ -2x_1 + x_2 + 2x_3,\ x_2 + 4x_4).$$

(ii) V is the vector space of polynomials over $\mathbb{R}$ of degree at most 2, and $T : V \to V$ defined by

$$T(p(x)) = 2p(x+1) - xp'(x) \qquad \forall p(x) \in V.$$

(iii) V is the vector space of all 2×3 matrices over $\mathbb{R}$, and $T : V \to V$ defined by

$$T(X) = \begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix} X \qquad \forall X \in V.$$