1. Let $u_1, u_2, \ldots, u_r$ and $v_1, v_2, \ldots, v_s$ and $e_1, e_2, \ldots, e_t$ be vectors in the vector space V over $\mathbb{R}$. Explain what is meant by each of the following statements.

   (i) $u_1, u_2, \ldots, u_r$ are linearly independent;

   (ii) $v_1, v_2, \ldots, v_s$ span V;

   (iii) $e_1, e_2, \ldots, e_t$ is a basis of V.

   Assuming that the vectors $v_1, v_2, \ldots, v_s$ span V, show that some subset of $\{v_1, v_2, \ldots, v_s\}$ is a basis of V.

   Stating clearly, but without proof, any preliminary results you need, explain why any two bases of V must have the same number of elements.

   Give an example of a vector space which is not spanned by a finite set of vectors. Briefly justify your answer.

2. Let U and V be subspaces of a finite dimensional vector space W. Prove that $U + V$ and $U \cap V$ are subspaces of W, and state (but do not prove) a formula which relates the dimensions of these subspaces to the dimensions of U and V.

   Now let $W = \mathbb{R}^4$ and

$$U = \{(a, b, c, d) : b + c + d = 0\} \quad \text{and} \quad V = \{(a, b, c, d) : b = -a\}.$$

   Find a basis of $U \cap V$ and calculate $\dim U$ and $\dim V$. Find a basis for $U + V$.

3. Let U and V be finite dimensional vector spaces over $\mathbb{R}$. What is meant by a linear map α from U to V?

   State, but do not prove, a formula relating the dimensions of the kernel of α and the image of α.

   Determine whether or not the following functions are linear maps from $\mathbb{R}^3$ to $\mathbb{R}^2$.

   (a) $\alpha((x, y, z)) = (x + z, -y + z)$

   (b) $\alpha((x, y, z)) = (x, y + 1)$

   (c) $\alpha((x, y, z)) = (x, 0)$

   (d) $\alpha((x, y, z)) = (xyz, 0)$.

   In the cases where α is a linear map, find bases for the kernel of α and the image of α.

4. (i) Construct a linear map α from $\mathbb{R}^3$ to $\mathbb{R}^3$ for which *precisely one vector* u in $\mathbb{R}^3$ satisfies $\alpha(u) = (0,1,0)$.

   (ii) Construct a linear map β from $\mathbb{R}^3$ to $\mathbb{R}^3$ for which *no vector* u in $\mathbb{R}^3$ satisfies $\beta(u) = (0,1,0)$.

   (iii) Construct a linear map γ from $\mathbb{R}^3$ to $\mathbb{R}^3$ for which *infinitely many vectors* u in $\mathbb{R}^3$ satisfy $\gamma(u) = (0,1,0)$.

   (iv) Explain why it is impossible to construct a linear map δ from $\mathbb{R}^3$ to $\mathbb{R}^3$ for which *precisely two vectors* u in $\mathbb{R}^3$ satisfy $\delta(u) = (0,1,0)$.

   (v) Construct a linear map θ from $\mathbb{R}^3$ to $\mathbb{R}^3$ for which some vector u in $\mathbb{R}^3$ satisfies $\theta(\theta(u)) \neq \mathbf{0}$ but every vector v in $\mathbb{R}^3$ satisfies $\theta(\theta(\theta(v))) = \mathbf{0}$.

   (vi) Construct a linear map ϕ from $\mathbb{R}^3$ to $\mathbb{R}^3$ for which some vector u in $\mathbb{R}^3$ satisfies $\phi(u) \neq u$ but every vector v in $\mathbb{R}^3$ satisfies $\phi(\phi(v)) = v$.

5. Let V be a vector space with basis $e_1, e_2, \ldots, e_n$ and let α be a linear map from V to V. What is meant by the matrix of α with respect to the given basis?

   Suppose that A is the matrix of α with respect to the basis $e_1, e_2, \ldots, e_n$ of V and B is the matrix of α with respect to the basis $f_1, f_2, \ldots, f_n$ of V. Prove that there exists an invertible matrix Q such that $B = Q^{-1}AQ$.

   What is meant by the rank of a matrix X? Find the ranks of X, Y and XY when

$$X = \begin{pmatrix} 3 & 0 & -2 \\ -1 & 1 & 1 \\ 1 & 2 & 0 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ -1 & 2 & 0 \end{pmatrix}.$$