1. Suppose that V is a vector space over $\mathbb{R}$, and that $v_1, ..., v_m, w_1, ..., w_n$ belong to V. Explain what is meant by the following statements.

$v_1, ..., v_m$ span V;
$w_1, ..., w_n$ are linearly independent.

(a) Prove that if $v_1, ..., v_m$ span V and $v_{m+1} \in V$ then $v_1, ..., v_m, v_{m+1}$ span V.

(b) Prove that if $w_1, ..., w_n$ are linearly independent and $n \geq 2$ then $w_1, ..., w_{n-1}$ are linearly independent.

(c) Suppose that v_1, v_2, v_3, v_4 span V. Does it follow that $v_1, v_2 - v_1, v_3 - v_2, v_4 - v_3$ span V? Justify your answer.

(d) Suppose that w_1, w_2, w_3, w_4 are linearly independent. Does it follow that $w_1 - 2w_2, 2w_2 - 3w_3, 3w_3 - 4w_4, 4w_4 - w_1$ are linearly independent? Justify your answer.

2. Let V be a finite dimensional vector space over $\mathbb{R}$ and let U and W be subspaces of V. Define the subspaces $U \cap W$ and $U + W$. Prove that

$$\dim U + \dim W = \dim(U \cap W) + \dim(U + W).$$

Suppose that U_1, U_2, U_3 are 5-dimensional subspaces of $\mathbb{R}^7$. Prove that

$$U_1 \cap U_2 \cap U_3 \neq \{\mathbf{0}\}.$$

3. Let V be a finite dimensional vector space over $\mathbb{R}$ and let α be a linear map from V to V.

Prove that α sends $\mathbf{0}$ to $\mathbf{0}$.

Define the kernel and image of α and show that they are subspaces of V.

State an equation relating the dimensions of V, Ker α and Im α. Give a very brief outline of how to justify this equation. (One or two sentences will suffice.)

Now let V be the vector space of polynomials of degree at most 3 with coefficients in $\mathbb{R}$. Suppose that α is a linear map from V to V and dim Ker $\alpha = 3$. Prove that Im $\alpha \subseteq$ Ker α or (Im $\alpha) \cap$ (Ker $\alpha) = \{\mathbf{0}\}$. Give examples to show that both possibilities can occur.

4. Let U and V be vector spaces over $\mathbb{R}$ and let α be a linear map from U to V. Suppose that $e_1, ..., e_n$ is a basis of U and $f_1, ..., f_m$ is a basis of V. Define the matrix of α with respect to these bases.

Now let $U = \mathbb{R}^2$, $V = \mathbb{R}^3$ and α be given by

$$\alpha : (x, y) \longmapsto (x + y, x + y, x + y)$$

(a) Find the matrix of α with respect to the bases

$$(1, 0), (0, 1) \text{ of } \mathbb{R}^2 \text{ and } (1, 0, 0), (0, 1, 0), (0, 0, 1) \text{ of } \mathbb{R}^3.$$

(b) Find the matrix of α with respect to the bases

$$(1, 2), (2, 3) \text{ of } \mathbb{R}^2 \text{ and } (0, 1, 1), (1, 1, 0), (3, 2, 0) \text{ of } \mathbb{R}^3.$$

(c) Find a basis e_1, e_2 of $\mathbb{R}^2$ and a basis f_1, f_2, f_3 of $\mathbb{R}^3$ such that the matrix of α with respect to these bases is

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

5. The entries a_{ij} in the $n \times n$ matrix $A = (a_{ij})$ are as follows.

$$a_{ij} = \begin{cases} -2 \text{ if } i = j \\ -4 \text{ if } j = i - 1 \\ -1 \text{ if } j = i + 1 \\ 0 \text{ otherwise.} \end{cases}$$

So, for example,

$$A_3 = \begin{pmatrix} -2 & -1 & 0 \\ -4 & -2 & -1 \\ 0 & -4 & -2 \end{pmatrix}.$$

Prove that for $n \geq 3$, we have

$$\det A_n = -2 \det A_{n-1} - 4 \det A_{n-2}.$$

Deduce that $\det A_n = 8 \det A_{n-3}$. Hence find formulae for $\det A_n$ when n has the form $3m - 1, 3m$ and $3m + 1$.

How is $\det A_n$ related to $\det(-A_n)$?